THE MATHEMATICAL GAZETTE

EDITED BY
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62 COLERAINE ROAD, BLACKHEATH, LONDON, S.E. 3.

LONDON
G. BELL & SONS, LTD., PORTUGAL STREET, KINGSWAY, W.C. 2

Vol. XXIV., No. 258. FEBRUARY, 1940. 3s. 6d. Net.

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Intending members are requested to communicate with one of the Secretaries (G. L. Parsons, Peckwater, Eastcote Road, Pinner, Middlesex; Mrs. E. M. Williams, 155 Holden Road, N.12). The subscription to the Association is 15s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette".

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VOL. XXIV. FEBRUARY, 1940. No. 258

REPORT OF THE COUNCIL FOR THE YEAR 1939.

Membership.

DURING the year 115 new members, including Junior members, have joined the Association. The membership is at present 1784, of whom 9 are honorary members, 1691 are ordinary members and 84 junior members. Thirty libraries are also on the roll. There have been a few resignations, mainly in consequence of war-time difficulties: the Council hopes that these members will be able to rejoin later. The Council records with regret the deaths of the following members of the Association: Mr. C. J. H. Barr (1902), Mr. H. C. Beaven (1906), Mr. F. C. Boon (1910), Prof. S. L. Loney (1896), Rev. J. J. Milne (1884), Mr. A. R. Prestwich (1904), Mr. J. Leadbeater (1934).

Branches.

The reports from the Branches show that in the early part of the year vigorous activity was being maintained. The outbreak of war and the consequent dispersal of many members caused the programmes of most branches to be abandoned or suspended, though several hope to recommence work in the spring. Three new branches have been formed, viz. the Sheffield and District Branch, the Plymouth and District Branch and the Nottingham and East Midland Branch (which has started since the war). Two university societies, the Oxford University Invariant Society and the King's College (London) Mathematical Society have been affiliated as Junior Branches. The Yorkshire Branch discussed the Spens Report at its summer meeting, and passed four resolutions criticising its recommendations; these were to have been considered by the Council at its October meeting. The Bristol Branch has been strengthened by members evacuated from London. The Manchester, South Wales and Midland Branches are at present out of action. The North-Eastern Branch has held a joint meeting with the Mathematics Section of the Conference of University and Secondary School

Teachers held at Durham, at which Mr. H. E. Dance, H.M.I., read a paper on "Optical Aids to Teaching", and at a later meeting Mr. A. Fletcher discussed the scope and content of the mathematics syllabus. The inaugural meeting of the Sheffield Branch was addressed by Prof. P. J. Daniell, who spoke on "Some doubts about mathematical teaching in the University"; at a later meeting Mr. A. Blackwell read a paper entitled "Some doubts about mathematical teaching in the Schools", which raised so many interesting points that a further meeting was held to continue the discussion. The Plymouth Branch has also discussed the Spens Report: an attractive programme for 1939-40 has been arranged, and the promoters hope to carry it through in spite of war difficulties. The Northern Ireland Branch also hopes to continue its programme as far as possible. The overseas branches are also continuing very useful activities, and every effort will be made to keep in touch with them. The Archimedeans (Cambridge Junior Branch) have produced two numbers of a journal called *Eureka*.

War-time difficulties have necessarily made the above report somewhat incomplete, and for the same reason the usual facts about the number of members and associates attached to each branch are not available.

The Mathematical Gazette.

The first three numbers of Vol. XXIII contained 336 pages; the last two numbers 160 pages. The reduction in scale was due to the outbreak of war, and the smaller size must be regarded as a war-time economy, by observance of which it is hoped that it will be possible to maintain publication of the customary five numbers per annum. Some sacrifice of material which would otherwise have appeared in the *Gazette* is implied: it has been decided that this sacrifice must be at the expense of the more advanced articles, so that contributions relating to school mathematics and reviews may suffer no undue compression. Since some branches have had to suspend activities, material which would have been put before Branch Meetings may possibly find its way into the *Gazette*.

Although there is to be no Annual Meeting in January 1940, the President has promised the *Gazette* an article for Vol. XXIV, which will take the place of the Presidential Address. Among other articles for Vol. XXIV, there is a detailed account of a series of school experiments on the teaching of multiplication, dealing in particular with the multiplication of decimals and the multiplication of money.

The Teaching Committee.

The Teaching Committee met in January 1939, a few days after the publication of the Spens Report. It decided to write to the Board of Education with regard to the comments on the teaching of mathematics contained in the Report. This letter appeared in the May issue of the *Gazette*. Further action was deferred until

members had considered the Report at their leisure. At this meeting a sub-committee was appointed to consider Sixth Form mathematics.

Meetings of sub-committees usually cease early in June through pressure of other duties, and during the interval until the autumn scripts for discussion at future meetings are prepared. Since it has not been practicable to hold any meetings this autumn, it is intended to publish in the *Mathematical Gazette*, from time to time, articles which might have been discussed in committee, and thus to keep members in touch with the work in progress.

The Sub-Committee on Trigonometry has nearly completed its discussions, and would now have been drafting its report for the press; the Sub-Committees on Technical Mathematics and on the Teaching of Mathematics to children up to the age of 11 are not quite so far ahead. The Examinations Sub-Committee has surveyed the various school certificate syllabuses, and felt that a definite policy should be framed. It was hoped to formulate this policy in time for a general discussion at the Annual Meeting in January 1940. The Sixth Form Committee has already discussed controversial points in the teaching of Calculus and Geometry.

The Library.

The demand for books remains almost constant from year to year; the service has been maintained, regulations having been framed to minimise the risks of loss on account of war.

Proposals for establishing the Library in a permanent home are under consideration and although negotiations have been suspended there is every reason to expect that they will be resumed to a satisfactory conclusion. A comprehensive programme of binding, including a mass of minor repairs and the binding of runs of journals, is essential to any long-range plan for the Library and this work, which will take about a year to complete, has been put in hand.

The Problem Bureau.

The number of applications has been rather above the normal, some 40 problems having been submitted during the past three months. Many of these were from members who had not applied previously: it is hoped that the Bureau will be of especial use to members who are working under difficult and abnormal conditions. In the earlier part of the year, applications were received from members in the Dominions, India and China.

Questions on Calculus and Mechanics continue to form the majority, together with a good many difficulties in Analytical Geometry.

General.

The rules as revised at the last General Meeting have been reprinted and circulated to all members.

Executive Council.

At the outbreak of war the Council, by a postal vote, resolved to appoint an Executive Committee, consisting of the President, the Treasurer, the Secretaries, the Librarian, the Editor of the *Mathematical Gazette*, the Chairman and Secretary of the Teaching Committee, together with Mr. A. W. Siddons, Mr. W. Hope-Jones and Miss M. Punnett. The Executive Committee decided to postpone the annual meeting arranged for January 1940, to modify the procedure for the delivery of *Gazettes* in the case of members on active service, and to provide for temporary suspension of membership in certain cases. The proposals of the Sub-Committee appointed to prepare plans for the future conduct of the Library were submitted to a vote of the Council and approved.

Officers and Council.

The Executive Committee has asked the President to continue in office for a further period, and Mr. Fletcher has expressed his willingness to do so. The other Officers were also asked to continue in their present offices.

The thanks of the Council and of the whole Association should once more be accorded to the Librarian, who has been engaged in negotiating for the future of the Library, and in arranging for necessary work to be put in hand. The Editor, whose excellent work in maintaining the *Gazette* is doing much to keep members of the Association together, should also be warmly thanked, as also should Mr. Gosset Tanner and the other members who have assisted him in the Problem Bureau.

M. HENRI FEHR.

It was in 1899 that *L'Enseignement Mathématique* was founded, and Henri Fehr, one of the two founders, is still inspiring and editing the journal. Forty years is a long time, and we congratulate our colleague and welcome the occasion to assure him of the esteem in which he is held and of the admiration which is felt for his work. It is not only that his journal has maintained the highest of professional standards, alike in discussing the teaching of classical parts of mathematics and in explaining the elements of contemporary developments. The first issue bore the words *Revue internationale*, and *L'Enseignement Mathématique* was adopted in 1909 as the official organ of the International Commission on the Teaching of Mathematics: M. Fehr was playing his part in creating the interlocked civilisation that the twentieth century in its first decade seemed to be promising. The dawn was false and for twenty-five years the light has been fading, but still the words *Revue internationale* blazon the faith which M. Fehr has not ceased to hold. We wish him many active years, and the happiness of seeing the daybreak.

SEGMENT LOCI.

By N. M. Gibbins.

In a lecture delivered to the London Branch on 19th October, 1935, and reported in the *Gazette*, Vol. XX, p. 233, Mr. C. V. Durell showed a large number of interesting loci suitable for boys and girls up to school certificate standard. It is proposed in this article to pursue the matter further for those of higher certificate standard.

1. If AB be a segment of a line, and perpendiculars be drawn to it through A and B, we may call the part of the plane between the perpendiculars the "region" of AB. Then if P be any point in the region, not on AB, the point on AB nearest to P is the foot of the perpendicular from P on AB; but if P be outside the region, the point nearest to P is A or B. It is in this sense that we shall speak of the "distance" of P from AB.

Quite a surprising variety of results follow from applying this idea to loci. For example, if we take two unequal segments AB, BC meeting at B at any angle, we may find the locus of a point which is equidistant from them. Part of the locus is the internal bisector

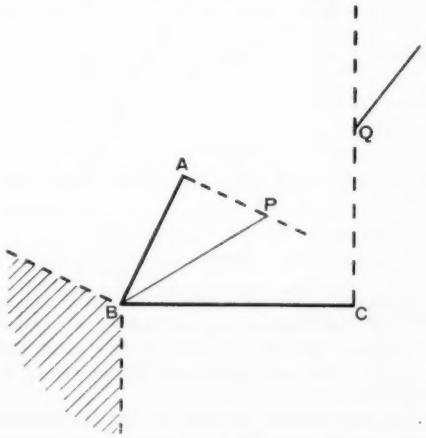

Fig. 1.

of ABC, from B to P, where $PAB = 90°$. Another part is that portion of the perpendicular bisector of AC which is to the right of QC, where $QCB = 90°$. Between P and Q the locus is an arc of the parabola having A as focus and BC as directrix, and touched by the former lines at P and Q respectively. Furthermore, every point in the shaded area is part of the locus.

If one segment BC has one end on another AD, as in Fig. 2, we get, below BC, BR, the bisector of DBC; RS, the parabola focus

D and directrix BC, and from S the perpendicular bisector of DC. Also BT, perpendicular to AD, is part of the locus, being the line common to the two " shaded areas " for AB, BC and DB, BC.

When the segments overlap, the locus consists of portions of the bisectors of the angle between them, four parabolas, and the four perpendicular bisectors of pairs of extremities.

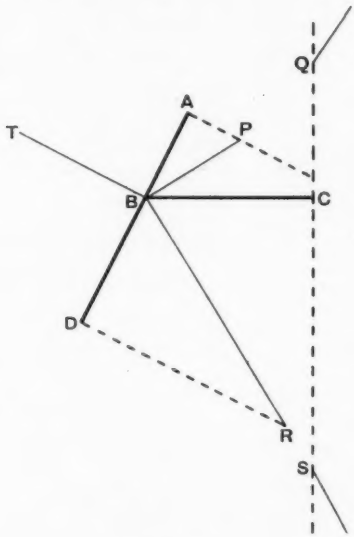

FIG. 2.

When the segments have no point in common and are coplanar we get a variety of cases depending on how each segment is related to the other's region. In Fig. 3 parts of the locus are similar to those in Fig. 2, and they are connected by an arc of the parabola focus A and directrix CD.

The locus of a point which moves so that its distance from the shorter segment is twice that from the longer one is shown in Fig. 4. On the left we have an arc of the ellipse focus A, directrix CD, eccentricity $\frac{1}{2}$; then two lines PQ and UT ; next QR and TS, parts of the two branches of the hyperbola focus D, directrix AB, eccentricity 2 ; and lastly part of the Apollonius circle for D, B. If the shorter segment is further shortened from the left, part of the locus may be an arc of the Apollonius circle for A, C separating two arcs of the ellipse. Each part of the locus touches the next part.

2. The principle can be extended to other relations between distances besides that of constant ratio. The construction of the

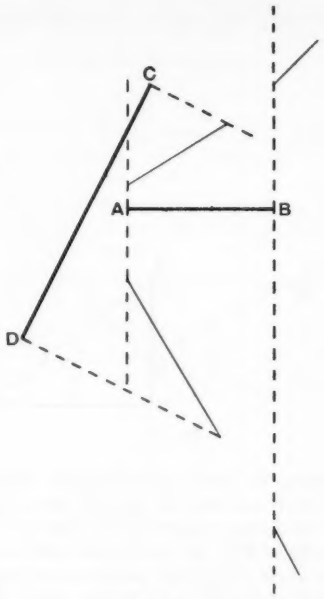

FIG. 3.

following table gives an opportunity for the rapid revision of a number of properties.

Relation.	Two Lines.	Line and Point.	Two Points.
$r_1 \pm r_2 = c$.	Line (equal intercepts).	Parabola.	Conic (foci).
$r_1^2 + r_2^2 = c^2$.	Ellipse (equi-conjugates).	Ellipse.	Circle.
$r_1^2 - r_2^2 = c^2$.	Conic.	Parabola.	Line.

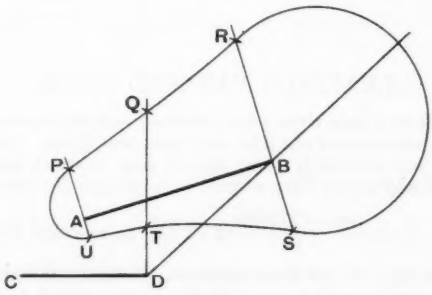

FIG. 4.

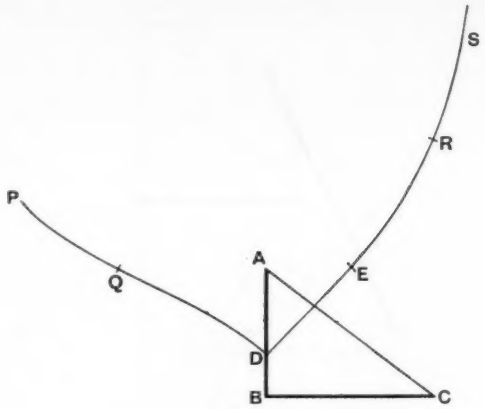

FIG. 5.

Fig. 5 shows the locus for two perpendicular segments of lengths six and eight units with one end of each in common, and such that the distance from the longer segment less that from the shorter is two units. It consists of PQ, the hyperbola having A and B as foci and major axis two units ; QD, the parabola having B as focus and a line parallel to AB and two units to the right as directrix ; the line DE ; ER, the parabola having A as focus and the line through D parallel to BC as directrix ; RS, the hyperbola having A and C as foci and major axis two units.

3. The consideration of what happens when the segments are not in the same plane belongs to the realm of university mathematics and may be left to others. In conclusion, I desire to acknowledge indebtedness to my colleague, Mr. T. A. J. Spencer, for putting this idea into my head, stopping me in time, and preparing the last two figures. N. M. G.

GLEANINGS FAR AND NEAR.

1301. " . . . Most of these titles were in German, and they were all concerned with terrifying abstractions of Light and Space and Time. And when you opened them, you invariably came upon a page in which some deathless drama of mathematics was being waged . . . something like this :

'*If we take* $a-b=\dfrac{bqz+\sqrt{870{\cdot}000}}{N}(2+2=5)$ *to represent speed at which light travels. . . .*'

And after reading two or three sentences, you put the book down, and thanked God that your feet were firmly on the ground. . . ."—Beverley Nichols, *A Village in a Valley*, 1934, p. 128. [Per Prof. J. P. Dalton.]

THEOREMS ON PERSPECTIVITY.

By R. T. Robinson.

I.

Any plane KLPQN cuts the edges of a tetrahedron ABCD in six points forming a complete quadrilateral. If UVW is the triangle formed by the diagonals of this quadrilateral, then

(i) *the triangular faces BCD, ACD, ... of the tetrahedron are each of them in perspective with the triangle UVW from points A_1, B_1, C_1, D_1 respectively ;*

(ii) *the tetrahedra ABCD, $A_1B_1C_1D_1$ are in perspective from U, V, W and from a fourth point Ω.*

1. In the triangles ABC, UVW (Fig. 1), BC, VW intersect at N, CA, UW at Q, AB, UV at P and R, Q, N are collinear. Hence AU, BV, CW intersect in a point, D_1 say. Similarly from the

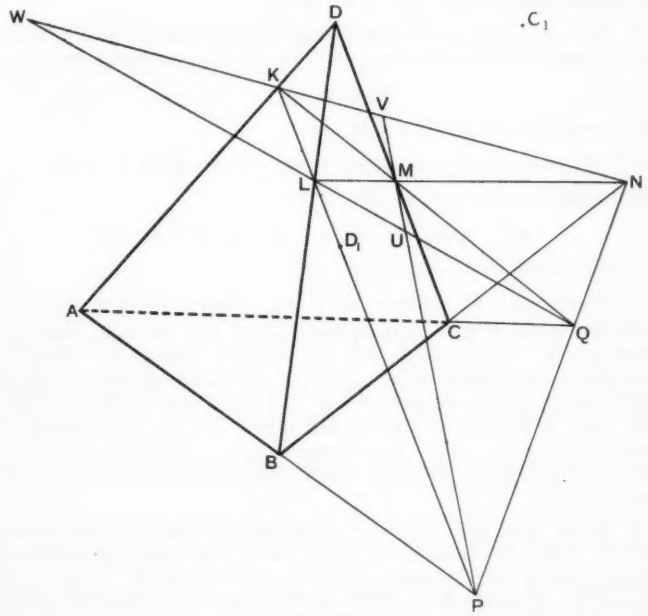

Fig. 1.

triangles BCD, UVW it can be shown that BW, CV, DU meet in a point A_1 ; from the triangles ACD, UVW that AW, CU, DV meet in a point B_1 ; and from the triangles ABD, UVW that AV,

BU, DW meet in a point C_1. Hence the triangles BCD, ACD, ... are in perspective with the triangle UVW from the points A_1, B_1, C_1, D_1 respectively.

2. From 1, AD_1 and A_1D intersect at U, and BC_1 and B_1C intersect at U. Hence the tetrahedra $ABCD$, $A_1B_1C_1D_1$ are in perspective from U, and, by a similar argument, from V and from W.

3. AD_1 and A_1D intersect at U. Hence DD_1 and AA_1 intersect. Similarly DD_1 and BB_1 intersect, and also DD_1 and CC_1, but BB_1 and CC_1 are not in the plane AA_1DD_1. Therefore AA_1, BB_1, CC_1, DD_1 meet in the same point, Ω say.

4. The plane AA_1DD_1 is the plane DKU, the plane BB_1DD_1 is the plane DLV, the plane CC_1DD_1 is the plane DMW. Hence DD_1 lies on the line joining D to the intersection of UK, LV, MW, say H_4. Similarly AA_1 lies on the line joining A to the intersection of UK, QV, PW, say H_1; BB_1 lies on the line joining B to the intersection of UN, LV, PW, say H_2; CC_1 lies on the line joining C to the intersection of UN, QV, MW, say H_3.

5. Taking $ABCD$ as the tetrahedron of reference, if the equation of the plane UVW is $\alpha/l+\beta/m+\gamma/n+\delta/p=0$, the coordinates of Ω are

$$(l, m, n, p);$$

the coordinates of A_1, B_1, C_1, D_1 are

$$(-l, m, n, p),\ (l, -m, n, p),\ (l, m, -n, p),\ (l, m, n, -p);$$

the coordinates of U, V, W are

$$(l, -m, -n, p),\ (l, -m, n, -p),\ (l, m, -n, -p);$$

the coordinates of H_1, H_2, H_3, H_4 are

$$(-3l, m, n, p),\ (l, -3m, n, p),\ (l, m, -3n, p),\ (l, m, n, -3p).$$

6. The point P $(l, -m, 0, 0)$ lies on AB and on A_1B_1, and the point Q $(l, 0, -n, 0)$ lies on AC and on A_1C_1, and so on. That is, the edges of the tetrahedron $A_1B_1C_1D_1$ pass through the six points of the complete quadrilateral KLM, ...; and it may be noticed that AB, A_1B_1, UV meet in a point and that CD, C_1D_1, UV meet in a point, and so on. Also, the triangles $B_1C_1D_1$, $A_1C_1D_1$, ..., are in perspective with the triangle UVW from the points A, B, C, D respectively.

7. When Ω is the centre of perspective of the two tetrahedra $ABCD$, $A_1B_1C_1D_1$, then UVW is the plane containing the points of intersection of corresponding edges of the tetrahedra; when U is the centre of perspective, ΩVW is the plane containing the points of intersection of the corresponding edges, and so on.

The vertices of the tetrahedra $ABCD$, $A_1B_1C_1D_1$, ΩUVW arrange themselves as in Fig. 2. They so arrange themselves that each pair of tetrahedra is in perspective from the four vertices of the remaining tetrahedron.

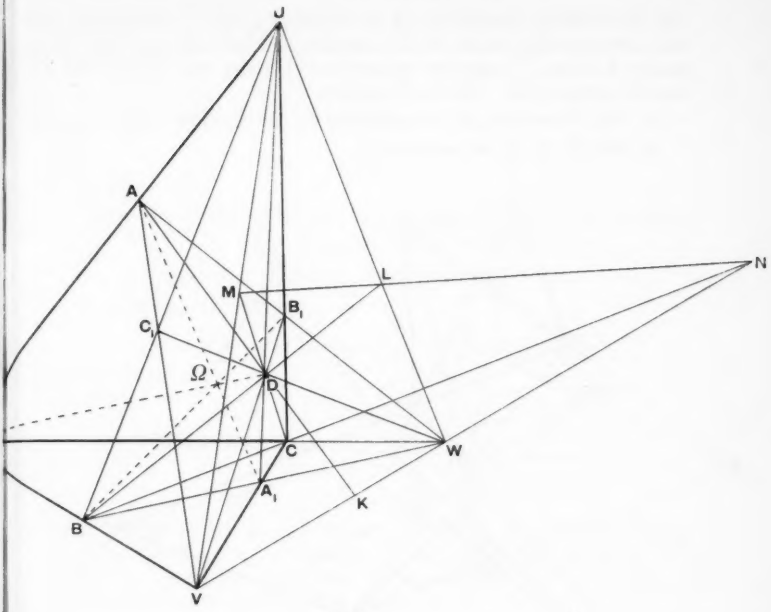

FIG. 2.

II.

If two tetrahedra $ABCD$, $A_1B_1C_1D_1$ are in perspective from a point O, the corresponding edges AB, A_1B_1, ... CD, C_1D_1, ... meet at points denoted by AB, ... CD, ..., say: these six points form a complete quadrilateral, and if U, V, W are the vertices of the triangle formed by the three diagonals of this quadrilateral, then by the last theorem

(i) *the tetrahedron $ABCD$ is in perspective with a tetrahedron $\alpha\beta\gamma\delta$, the centres of perspective being U, V, W and a point Ω ;* *

(ii) *the tetrahedron $A_1B_1C_1D_1$ is in perspective with a tetrahedron $\alpha_1\beta_1\gamma_1\delta_1$, the centres of perspective being U, V, W and a point Ω_1.*

The edge $\alpha\beta$ of the tetrahedron $\alpha\beta\gamma\delta$ meets the edge AB of the tetrahedron $ABCD$ in the point AB, that is, the point of intersection of AB, A_1B_1, and there is a similar result for each edge of the tetrahedron $\alpha\beta\gamma\delta$; in the same way it can be seen that the edge $\alpha_1\beta_1$ of the tetrahedron $\alpha_1\beta_1\gamma_1\delta_1$ also passes through the point

* In Fig. 3, γ is off the paper.

AB, with similar results for all the edges of $\alpha_1\beta_1\gamma_1\delta_1$. Therefore since the corresponding edges of the tetrahedra $\alpha\beta\gamma\delta$, $\alpha_1\beta_1\gamma_1\delta_1$ meet in six points forming a complete quadrilateral, these two tetrahedra are also in perspective. It can be shown

(i) *that O is a centre of perspective of the two tetrahedra $\alpha\beta\gamma\delta$, $\alpha_1\beta_1\gamma_1\delta_1$;*

(ii) *that Ω, O, Ω_1 are collinear.*

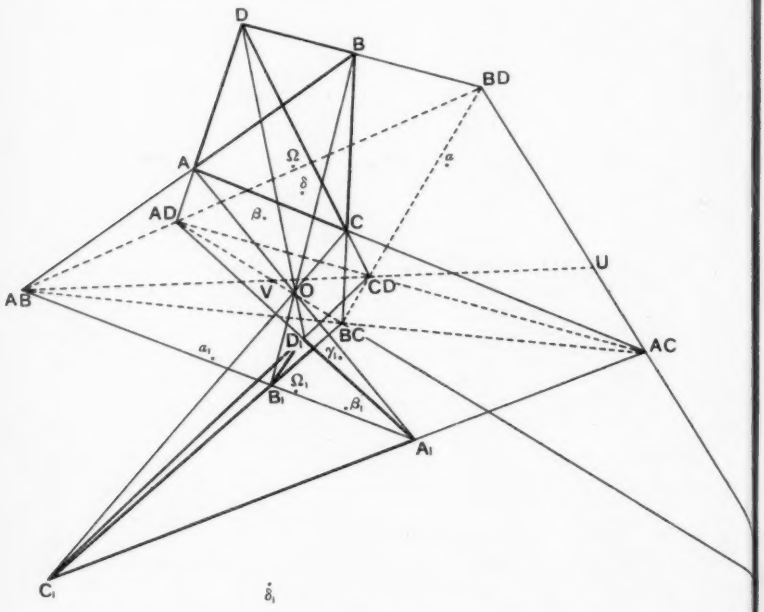

FIG. 3.

1. If $ABCD$ is the tetrahedron of reference, and if the coordinates of O are $(\alpha_1, \beta_1, \gamma_1, \delta_1)$, the coordinates of A_1 can be put in the form $(x_1\alpha_1, \beta_1, \gamma_1, \delta_1)$, those of B_1 in the form $(\alpha_1, x_2\beta_1, \gamma_1, \delta_1)$, and so on. The coordinates of the point AB, that is, of the intersection of AB and A_1B_1, are $\{\alpha_1(1-x_1), -\beta_1(1-x_2), 0, 0\}$, those of CD are $\{0, 0, \gamma_1(1-x_3), -\delta_1(1-x_4)\}$, and so on. The equation of the plane containing the complete quadrilateral $AB \ldots CD \ldots$ is

$$\frac{\alpha}{\alpha_1(1-x_1)}+\frac{\beta}{\beta_1(1-x_2)}+\frac{\gamma}{\gamma_1(1-x_3)}+\frac{\delta}{\delta_1(1-x_4)}=0.$$

Hence from the preceding theorem we obtain the coordinates of Ω, U, ... , where after giving the coordinates of Ω the others can be specified simply by the variations in sign :

Ω	$\alpha_1(1-x_1)$	$\beta_1(1-x_2)$	$\gamma_1(1-x_3)$	$\delta_1(1-x_4)$
U	+	−	−	+
V	+	−	+	−
W	+	+	−	−
α	−	+	+	+
β	+	−	+	+

and so on.

2. Since the vertex δ_1 of the tetrahedron $\alpha_1\beta_1\gamma_1\delta_1$ is the point of intersection of UA_1, VB_1, WC_1, the coordinates of δ_1 are

$$\alpha_1(1+x_1),\ \beta_1(1+x_2),\ \gamma_1(1+x_3),\ \delta_1(3-x_4)\,;$$

for the coordinates of U are

$$\alpha_1(1-x_1),\ -\beta_1(1-x_2),\ -\gamma_1(1-x_3),\ \delta_1(1-x_4),$$

the coordinates of A_1 are $x_1\alpha_1,\ \beta_1,\ \gamma_1,\ \delta_1$,
and we have

$$\begin{aligned}
\alpha_1(1-x_1)+2x_1\alpha_1 &= \alpha_1(1+x_1),\\
-\beta_1(1-x_2)+2\beta_1 &= \beta_1(1+x_2),\\
-\gamma_1(1-x_3)+2\gamma_1 &= \gamma_1(1+x_3),\\
\delta_1(1-x_4)+2\delta_1 &= \delta_1(3-x_4),
\end{aligned}$$

that is, the point whose coordinates are those on the right of these equations lies on UA_1. Similarly it can be shown to lie on VB_1 and WC_1. Therefore these are the coordinates of the vertex δ_1. Similarly the coordinates of the vertex α_1 are

$$\alpha_1(3-x_1),\ \beta_1(1+x_2),\ \gamma_1(1+x_3),\ \delta_1(1+x_4),$$

and so on.

3. The straight lines joining the vertices $\alpha,\ \alpha_1$ of the tetrahedron $\alpha\beta\gamma\delta,\ \alpha_1\beta_1\gamma_1\delta_1$ passes through O, for the coordinates of α are

$$-\alpha_1(1-x_1),\ \beta_1(1-x_2),\ \gamma_1(1-x_3),\ \delta_1(1-x_4),$$

and

$$\begin{aligned}
\alpha_1(3-x_1)-\alpha_1(1-x_1) &= 2\alpha_1,\\
\beta_1(1+x_2)+\beta_1(1-x_2) &= 2\beta_1,
\end{aligned}$$

and so on.

Similarly it can be shown that the straight lines $\beta\beta_1,\ \gamma\gamma_1,\ \delta\delta_1$ each pass through O. Hence the tetrahedra $\alpha\beta\gamma\delta,\ \alpha_1\beta_1\gamma_1\delta_1$ are in perspective from O.

4. The coordinates of A_1 being $(x_1\alpha_1,\ \beta_1,\ \gamma_1,\ \delta_1)$ and the coordinates of the vertex α_1 being $\alpha_1(3-x_1),\ \beta_1(1+x_2),\ \gamma_1(1+x_3),\ \delta_1(1+x_4)$, then since

$$\begin{aligned}
2x_1\alpha_1+\alpha_1(3-x_1) &= \alpha_1(3+x_1),\\
2\beta_1+\beta_1(1+x_2) &= \beta_1(3+x_2),\\
2\gamma_1+\gamma_1(1+x_3) &= \gamma_1(3+x_3),\\
2\delta_1+\delta_1(1+x_4) &= \delta_1(3+x_4),
\end{aligned}$$

the point Ω_1, whose coordinates are those on the right of these equations, lies on $A_1\alpha_1$, and similarly it can be shown to lie on $B_1\beta_1$, $C_1\gamma_1$,

$D_1\delta_1$. Thus the tetrahedra $A_1B_1C_1D_1$, $\alpha_1\beta_1\gamma_1\delta_1$ are in perspective from Ω_1.

5. The coordinates of Ω being $\{\alpha_1(1-x_1), \ldots\}$ and those of Ω_1 being $\{\alpha_1(3+x_1), \ldots\}$, then since $\alpha_1(1-x_1)+\alpha_1(3+x_1)=4\alpha_1$, and so on, the point O, whose coordinates are α_1, β_1, γ_1, δ_1 lies on $\Omega\Omega_1$.

6. It can be shown that if AOA_1 meets the plane BCD in S, then the coordinates of A_1, which have been taken as $(x_1\alpha_1, \beta_1, \gamma_1, \delta_1)$ can be put in the form

$$\{(A_1S \,.\, OA/SO \,.\, AA_1)\,\alpha_1, \beta_1, \gamma_1, \delta_1)\}$$

whence $$x_1 \equiv (AOSA_1).$$

When $x_1 = -1$, then $(AOSA_1) = -1$.

7. When $x_1 = x_2 = x_3 = x_4 = -1$, the vertices α, β, γ, δ of the tetrahedron $\alpha\beta\gamma\delta$ coincide respectively with A_1, B_1, C_1, D_1, and the vertices α_1, β_1, γ_1, δ_1 of the tetrahedron $\alpha_1\beta_1\gamma_1\delta_1$ coincide respectively with A, B, C, D, and each of the points Ω, Ω_1 coincides with O.

R. T. R.

1302. *A "proof" that* $1=0$.

$$\sum_1^\infty \frac{x^n - x^{n-1}}{(1+x^n)(1+x^{n-1})} = \sum_1^\infty \left[\frac{1}{1+x^{n-1}} - \frac{1}{1+x^n}\right]$$

$$= \frac{1}{1+1} - \frac{1}{1+x} + \frac{1}{1+x} - \frac{1}{1+x^2} + \frac{1}{1+x^2} - \ldots$$

$$= \tfrac{1}{2} - \left(\frac{1}{1+x} - \frac{1}{1+x}\right) - \left(\frac{1}{1+x^2} - \frac{1}{1+x^2}\right) - \ldots$$

$$= \tfrac{1}{2}$$

$$\sum_1^\infty \frac{x^n - x^{n-1}}{(1+x^n)(1+x^{n-1})} = \sum_1^\infty \left[-\frac{x^{n-1}}{1+x^{n-1}} + \frac{x^n}{1+x^n}\right]$$

$$= -\frac{1}{1+1} + \frac{x}{1+x} - \frac{x}{1+x} + \frac{x^2}{1+x^2} - \frac{x^2}{1+x^2} + \ldots$$

$$= -\tfrac{1}{2} + \left(\frac{x}{1+x} - \frac{x}{1+x}\right) + \left(\frac{x^2}{1+x^2} - \frac{x^2}{1+x^2}\right) + \ldots$$

$$= -\tfrac{1}{2}.$$

Thus $\frac{1}{2} = -\frac{1}{2}$ and so $1=0$. [Per Prof. J. P. Dalton.]

1303. ORTHOGONAL TRANSFORMATION ON THE SPHERE!

"It is said that Viceroys were expected to keep their watches in time with two latitudes."—Shane Leslie, *Studies in Sublime Failure*, 1932, p. 220. [Per Prof. J. P. Dalton.]

1304. No one, I think, could fail to be aware of a feeling of luxury in talking to a fine scholar. It is like wearing fine linen next the skin instead of horse-hair. A knowledge of the differential calculus, on the other hand, seems to me to make very little difference to the comfort of an ordinary human contact.—Lady Violet Bonham-Carter, *John O'London's Weekly*, July 29, 1938, p. 630. [Per Mr. T. S. Venkatraman.]

ABSTRACT ALGEBRA.

By A. R. Richardson.

The algebra taught at present in schools and colleges and discussed in recent reports is essentially that of Euler and Chrystal as regards both content and method. Must this tradition continue? It is hoped that this short note will suggest the contrary and indicate the possibility of incorporating the ideas and methods of modern algebra in elementary teaching. To this end attention will be drawn to some of the simpler concepts of the subject.

Modern algebra is *abstract*,* *i.e.* its symbols are undefined, and their laws of combination, which are determined by suitable postulates, are therefore the sole objects of study. The resulting abstract methods and theorems are applicable to any special system required for particular problems, and, conversely, methods and theorems in different mathematical domains, *e.g.* algebra or geometry, can often be recognised to be special cases of the same abstract mathematics. In short, the various theorems of ordinary mathematics may be derived, once and for all, from a few axioms which are the same for all branches of the subject. For example, the theory of algebraic varieties and that of ideals in commutative fields are in essence the same although one is expressed in geometric terms and the other in those of algebra.

A further important argument in favour of abstract methods is that they exclude all but the vital facts and processes. Assumptions must be stated explicitly, and hence such unconscious hypotheses as are often made when dealing with special systems and also the various irrelevancies, which frequently occur when arguments are based on the technique of special systems, are either avoided or may be recognised.

These considerations alone would justify the abandonment of the opposite doctrine that the external world is the source of all mathematical inspiration, but there are other reasons which are more fundamental. The enormous growth of mathematical knowledge in ever widening fields demands that coordination and outlook which is a feature of abstraction. Further, the abstract method enables the subject to be considered as an organic whole. Finally, abstraction makes another strong appeal to aesthetic taste by its solution of various problems of completeness, *e.g.* the determination of all algebraic systems with given properties such as those in which the Galois theory is valid or in which factorisation is unique.

As it is impossible to discuss here the work of Hilbert, Gödel and others on symbolic logic, we shall therefore commence with a set, S, of symbols or elements, noting that many of the most important theorems of mathematics deal, not with the symbols themselves, but with relations between sets of symbols. For example, in

* To some extent all mathematics is abstract as the symbols are usually undefined and, as pointed out by F. Klein, the existence of a calculating machine shows that we deal not with integers but primarily with their laws of combination.

geometry, although points are often regarded as the elements, the results relate to properties of sets of points, planes, surfaces, neighbourhoods, and so on. Note that this classification of elements into sets is not peculiar to mathematics but is common to all sciences including that of language.

Congruence. Let a set, S, be divided into classes by some relation, R, and for the present assume that these classes are mutually exclusive. Denote the fact that a and b are in the same class by $a \equiv b \pmod{R}$. Evidently

(i) either $a \equiv b$ or $a \not\equiv b$;
(ii) $a \equiv a$;
(iii) if $a \equiv b$ then $b \equiv a$;
(iv) if $a \equiv b$ and $b \equiv c$ then $a \equiv c$.

A relation, R, of this type is termed an *equivalence relation* (5) and is a special case of an *ordering relation*, O, for which

(i) $a \subset a$; (ii) if $a \subset b$ and $b \subset c$ then $a \subset c \pmod{O}$.

In an ordering relation it does not follow that there is a relation (mod O) between every pair of elements a and b.

A set over which an ordering relation is defined is termed an *ordered set*, and if for every pair of elements a and b either $a \subset b$ or $b \subset a$ the set is said to be *simply ordered*.

Conversely, any sub-division of a set into mutually exclusive classes determines an equivalence relation.

In a set over which an equivalence relation is defined "equals" may be substituted for "equals" without altering the ordering relation ("things which are equal to the same thing are equal to one another"). If "equals" may be substituted for "equals", the set may be ordered, *i.e.* a symbol $\sim$ such that $a \sim a$, $b \sim b$, $c \sim c$, and if $a \sim b$ and $c \sim b$ then $a \sim c$ defines an equivalence relation.

(*a*) *Examples of ordering relations:*
(i) in integers if a divides b then $a \subset b$,
(ii) in real numbers if $a < b$ then $a \subset b$,
(iii) in sets if a is a sub-set of b then $a \subset b$,
(iv) in algebraic fields if A is algebraic over B and B algebraic over C then A is algebraic over C (1).

(*b*) *Examples of equivalence relations:*
(i) in integers $a \equiv b \pmod{R_p}$ if $a = b + \lambda p$. This is the ordinary congruence of the textbooks,
(ii) in projective geometry $a \equiv b$ if a is projective with b,
(iii) similarity and parallelism in geometry,
(iv) in group theory conjugate elements are equivalent.

Not all mathematical relations are ordering, *e.g.* "being perpendicular to", "being the square of", are not ordering relations.

The set of all ordering relations over a given set may be itself ordered. Let $O \subset O'$ (mod P) if $a \subset b$ (mod O) implies that $a \subset b$ (mod O'). Evidently $O \subset O$ (mod P), and if $O \subset O'$ (mod P) and $O' \subset O''$ (mod P) then $O \subset O''$ (mod P), *i.e.* P is an ordering relation.

The following remarks assume the ordering of the set of ordering relations of a set (mod P). They may be ordered in other ways.

Structures (17). If for any two elements a and b of the ordered set D and M exist such that

$$a \subset M \text{ and } b \subset M\,; \text{ if } a \subset c \text{ and } b \subset c \text{ then } M \subset c\,;$$

$$D \subset a \text{ and } D \subset b\,; \text{ if } c \subset a \text{ and } c \subset b \text{ then } c \subset D.$$

D and M are termed the *cross-cut* and *union* respectively and written $D = (a, b)$ and $M = [a, b]$ and the set is termed a *structure*.

A *principal chain* (17) $a \subset a_2 \subset a_3 \ldots \subset a_n = b$ *between* a and b is one in which it is impossible to insert other elements without altering the ordering relation and then a_{i+1} is said to be *prime over* a_i or a_{i+1} *covers* a_i (17) ; (5).

The set of all ordering relations ordered (mod P) is a structure in which $D = (O_i, O_j)$ if and only if $a \subset b$ (mod O_i) and $a \subset b$ (mod O_j), whilst $M = [O_i, O_j]$ is such that $a \subset b$ (mod M) if $u_1, u_2, \ldots u_{2s-1}$ exist such that $a \subset u_1$ (mod O_i), $u_1 \subset u_2$ (mod O_j), $u_2 \subset u_3$ (mod O_j), $\ldots$, $u_{2s-1} \subset b$ (mod O_j).

The set of all equivalence relations is a sub-structure of the set of all ordering relations, the cross-cut and union being defined as above.

Other modes of composition of equivalence relations are possible. For example, over a finite set of elements, if the classes (mod R_λ) contain $\{\lambda\} = \{n_{11}, n_{12}, \ldots n_{1s}\}$ elements respectively, and if the classes (mod R_μ) contain $\{\mu\} = \{n_{21}, n_{22}, \ldots n_{2r}\}$ elements respectively, then (15)

$$R_\lambda \times R_\mu = \sum_\nu g_{\lambda\mu\nu} R_\nu,$$

where the $g_{\lambda\mu\nu}$ are positive or zero integers and Σ has the logical meaning "and". These relations are of fundamental importance in quantum mechanics, and this mode of looking at the matter shows that the physical properties depending on the numbers, $g_{\lambda\mu\nu}$, relate merely to a distribution of the elements in classes and their combination into other classes.

Examples of structures :

(i) in projective geometry the cross-cut of two elements is their intersection whilst their union is the least dimensional element containing both (5) ;

(ii) in integers or polynomials in commutative algebra M and D are the L.C.M. and H.C.F. respectively ;

(iii) the sub-groups of a group and the sub-rings of a ring (1) ;

(iv) in sets $(a, b) = a \,.\, b$ and $[a, b] = b \dotplus a$ (12).

At this point it is convenient to draw attention to the fact that a class of elements, when subjected to suitably postulated laws of combination, is regarded as an element of a new number system, *e.g.* :

(i) The integers may be regarded as classes each containing two elements of the primitive integers 1, 2, 3, ... ; the rational fractions as classes of integers ; the real numbers as classes of rational fractions, and so on.

(ii) In statistics numbers are put into sets to yield frequency distributions.

Correspondences. The relation, C, between the elements, a, of a set S and the class, $\bar{a}$, to which a belongs (mod C) regarded as an element of a new system $\bar{S}$ is an example of a type of relation between sets termed a correspondence (1). That is, to every element a of S there corresponds a unique element $\bar{a}$ of $\bar{S}$, and every $\bar{a}$ of $\bar{S}$ is the correspondent of at least one element a of S. We write

$$C : S \to \bar{S} \; ; \; \bar{S} = \frac{S}{C} \text{ and also } S \sim \bar{S}.$$

Given two correspondences $C_1 : S \sim \bar{S}$; $C_2 : \bar{S} \sim \bar{\bar{S}}$, then there exists a correspondence, denoted by C_1C_2, such that $S \sim \bar{\bar{S}}$. Multiplication so defined is associative, *i.e.* $(C_1C_2)C_3 = C_1(C_2C_3)$.

A correspondence may be established between a certain set of *groupoids*, *i.e.* systems closed to a binary operation, termed multiplication, and a set of ordering relations over a set. The greater part of the mathematics described in this paper may be discussed by using the above ideas (9). The following presentation however is simpler as it involves ideas which are more familiar to the reader.

The set of all correspondences over a set is an associative groupoid and the set of all (1, 1) correspondences is a sub-group of this, a (1, 1) correspondence being one in which $S \sim S'$ and $S' \sim S$.

Correspondences between sets or between elements of the same set may be determined in various ways. In a number system closed to a binary operation, termed multiplication, (1, 1) correspondences may sometimes be defined algebraically, *e.g.* in a system in which $ax = a'$ has a unique solution x when a and a' are given and a unique a' when a and x are given, then $ax = a'$ defines C_x: $a \rightleftarrows a'$. Further, if the multiplication is associative, C_xC_y: $ax = a'$, $a'y = a''$, *i.e.* $(ax)y = a \,.\, xy = a''$, *i.e.* $C_xC_y = C_{xy}$. In other words, the correspondences afford a *representation* of the set (8).

Examples of correspondence :

(i) $a = f(a')$, the idea of function ;

(ii) $f(a) \to f'(a)$, $f'(a)$ being the derivative of a.

Examples of (1, 1) *correspondence :*

(i) $ax = a'$ as above ;

(ii) $a' = (\alpha a + \beta)/(\gamma a + \delta)$ in commutative fields or in geometry ;

(iii) $a = \log a'$;

(iv) $a \perp a'$ in projective geometry, a and a' being geometrical figures ;

(v) the correspondence between a point A on a surface and the point A' to which it is carried by a deformation of the surface without tearing. This is the starting-point of the group theory of geometry and of the topology associated with the connectivity of surfaces ;

(vi) correspondence between the elements of all principal chains between a and b in a Dedekind structure (17).

Examples of representation :

(i) Cayley's theorem that every finite abstract group may be represented by a group of permutations ;

(ii) the representation of an algebra by an algebra of matrices (8).

Both of these arise from the correspondence $Cx : ax = a'$. The correspondence itself gives rise to the theory of matrices and determinants.

In number systems S and S' over which binary operations are defined, which need not be the same, there may exist special types of correspondence (1), viz. $a \to a'$, $b \to b'$ may also imply that if $a'.b' = c'$ in S' then $a.b = c$ in S. In this case S is said to be *homomorphic* with S', written $S \simeq S'$. If in addition the correspondence is (1, 1), S is said to be *isomorphic* with S', written $S \cong S'$. An isomorphic correspondence between elements of the same set is termed an *automorphism*, and one of the most fundamental theorems in mathematics is :

The set of all automorphisms of a groupoid is a group.

Its importance lies in the fact that the relations between a groupoid and its sub-groupoids are determined by the structure of its automorphisms.

The binary operation referred to above may of course be the union or cross-cut of an ordered set. If $S \simeq S'$ with regard to both union and cross-cut, the sets will be said to be *structurally isomorphic* (17). A certain sub-set of the set of all equivalence relations over a set, S, may be ordered so as to be structurally isomorphic with S.

Examples :

(i) The set of all ordering relations of a set may be ordered so as to be homomorphic with a sub-set, viz. the sub-set of equivalence relations ;

(ii) in the Galois theory of equations (1) over an algebraic field there exists a structural isomorphism between the sub-fields of a normal extension, $\overline{N}$, of a field N and the sub-groups of the group of automorphisms of $\overline{N}$ with respect to N ;

(iii) in positive integers the set of numbers is isomorphic with set of their logarithms, the corresponding operations being of course multiplication and addition respectively ;

(iv) in groups an abstract group is simply isomorphic with a group of permutations (Cayley's theorem) ;

(v) an *evaluation* (1), (29) is a homomorphism between a structure and a set of real numbers in which multiplication in the structure corresponds to addition of reals.

These ideas may also be traced in other sciences, *e.g.* a correspondence may be established between the various compounds of the halogens.

It is but a short step now to the theory of *invariance* which deals with the elements of a set which are self-corresponding or which are unchanged by a group of automorphisms.

Examples of invariance :

(i) Invariance under the (1, 1) correspondence

$$w = (\alpha z + \beta)/(\gamma z + \delta)$$

leads to the theories of the elliptic (10) and automorphic functions (11) and to non-euclidean geometry (7).

(ii) Invariance under $ax = a$ leads to the ordinary theory of invariants and to the theory of integral equations.

(iii) Zeros, units, idempotents and other regular elements in an algebra (8).

(iv) Cross-ratio under a projective group.

(v) Connectivity of a surface under the group of deformations.

(vi) The set of integral curves under a Lie group of a differential equation.

(vii) The length of a principal chain between two elements in a Dedekind structure and the number of prime factors of a number in an integrally closed ring.

(viii) Symmetrical figures such as the regular solids are invariant under groups of rotations, reflexions, etc. ; other groups leave a triangle invariant (32).

(ix) Classes of conjugate elements in a group have the group characters as invariants (15).

(x) The Betti numbers under topological transformations.

Special structures. Most of the sets dealt with in mathematics are structures, and in order to obtain theorems about the reduction of such sets to simpler parts it is necessary to impose further restrictions. For example, the factorisation of integers is into primes ; the decomposition theorems about groups refer to normal subgroups ; the unique factorisation of ideals is into prime ideals. In all such cases the structure satisfies the *Dedekind* condition (2), (17).

If $A \subset C$ then $(C, [A, B]) = [A, (B, C)]$, ...(3)

or, in self-dual form,

$$[(A, B), (C, [A, B])] = ([A, B], [C, (A, B)]). \quad \ldots\ldots\ldots\ldots(4)$$

In a structure having only a finite number of elements (3) is the necessary and sufficient condition that all principal chains between any two elements have the same length.

The structures of ordering and equivalence relations (mod P) are not Dedekind structures, but a sub-set of the set of equivalence relations may be determined so as to be Dedekindian (9). The great importance of the Dedekind structures lies in the following fact :

Let the set of all elements C such that $A \supset C \supset B$ be the elements of a new number system, termed the *quotient-structure*, $\mathcal{A} = A/B$. Then in a Dedekind structure

$$[A, B]/A \cong B/(A, B), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where $\cong$ denotes that the isomorphism is structural, and conversely, if (5) and either the descending or ascending chain condition holds, the structure is Dedekindian.

This is the *second law of isomorphisms* of groups and ideals, and is the basis of nearly all work on structures.

The quotients are made into a structure and a law of multiplication is defined amongst certain elements by (17) : if $\mathcal{A} = A/B$ and $\mathcal{B} = B/C$, then $\mathcal{A} \times \mathcal{B} = A/C$.

In a Dedekind structure, if an element can be represented as the union of a finite number of indecomposable elements, then any two such representations contains the same number of components (17). Many decomposition theorems, however, are independent of the Dedekind condition (31).

The important part played by the theorem of Jordan-Hölder in the theory of equations is well known. It is a special case of the following (19) :

If a quotient, $\mathcal{A}$, can be represented in two ways as the product of prime quotients $\mathcal{A} = \mathcal{B}_1 \times \mathcal{B}_2 \times \ldots \times \mathcal{B}_s = \mathcal{C}_1 \times \mathcal{C}_2 \times \ldots \times \mathcal{C}_r$, then both factorisations have the same number of factors and these are similar, in the sense of the second law of isomorphism, in pairs. Further, if a quotient can be factorised in two ways as the product of quotients, not necessarily prime, then these factors may be further factorised so as to have the same number of factors similar in pairs.

Most theorems on the decomposition of ideals are independent of the existence of a multiplication, and it is only in integrally closed rings, such as the complex numbers, that these theorems have a multiplicative form.

Examples :

(i) In real numbers, ordered by " less than ", the interval $a \leqslant x \leqslant b$ is the quotient b/a.

(ii) The fundamental theorem of arithmetic is a special case of the theorem of Jordan-Hölder, and it should be possible to prove it without using the additive properties.

(iii) In integers $[a, b]/a = b/(a, b)$.

(iv) In groups, if A, B represent normal sub-groups, then

$$[A, B]/A \cong B/(A, B).$$

(v) In ideals in a commutative ring $[a, b]/a \cong b/(a, b)$.

(vi) In integers, if $a \subset b$ when a/b, then all principal chains of division are of the same length and, apart from the order, the quotients are the prime factors of b/a.

The abstract theorems, of which the theorems of Schmidt-Remak and Kurosch are special cases, and which relate to the representation of a group M as the *direct* product of *direct* (**18**) indecomposable components A, B, *i.e.* $M = [A, B]$, $(A, B) = E$ the unit element of the group, require further restrictive conditions relating to the lengths of chains. Most special algebraic systems satisfy still more restrictive conditions. For example, in theorems relating to Abelian groups and in the general theory of ideals in commutative rings with unique factorisation, the structure is *distributive*, *i.e.*

$$(A, [B, C]) = [(A, B), (A, C)] \text{ (17)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Another type of structure is the *complemented structure* (**22**), (**17**), in which each element A has a complement $\overline{A}$, which need not be unique, such that $(A, \overline{A}) = E$ and $[A, \overline{A}] = 0$, E and O being the unit and all elements respectively, *i.e.* $E \subset A \subset O$ for all A.

An example is the structure of the equivalence relations over a set.

A distributive, complemented structure is a *Boolean Algebra* (**25**), *i.e.* a ring with a unit in which every element is idempotent. The well-known algebra of logic is an example. These structures have applications to logic, to the calculus of classes and to the theory of topological spaces.

In a complemented Dedekind structure all descending chains have a finite length, and if, except for E and O defined above, the complement of any element is *not* unique, we arrive at the axioms of *projective geometry* (**14**), and an additional parallel axiom gives *affine geometry* (**14**). The advantage of this mode of treating geometry is that points, lines, surfaces, neighbourhoods, etc., appear symmetrically, and the special importance of the point disappears.

Duality (**20**). Referring back to the Dedekind condition (**4**), we note that cross-cut and union are interchangeable. This duality is the source of much of the reciprocity noted in mathematics.

Examples :

(i) The theorems of Jordan-Hölder and Schmidt-Remak are self-dual.

(ii) The L.C.M. and H.C.F. in integers or in polynomials in a commutative ring.

(iii) The symmetric functions a_n, h_n of the roots of an equation (**13**).

(iv) Conjugate elements in projective geometry.

(v) The various theorems of reciprocity in the theory of numbers.
(vi) Geometrical reciprocity of point and plane.
(vii) Duality of residuation and multiplication in structures (31).
(viii) Conjugate partitions and group characters (15).
(ix) Reciprocal functions.
(x) Centre and anti-centre in groups.
(xi) The Betti numbers of dimension k are equal to those of dimension $(n-k)$.

In the early part of this note a set was separated into mutually exclusive classes of elements. If this restriction is abandoned, but if instead the elements in a class are ordered, we get the theory of *vectors*. With vectors the correspondence $ax=a'$ leads to the theory of matrices and determinants ; and to the theory of integral equations and orthogonal functions.

Lack of space forbids a discussion of other aspects of abstract algebra ; for example, the problems of completeness (16) already mentioned above, and the problems of determining to what extent a given theorem, *e.g.* that of Jordan-Hölder, is true for particular systems (23).

This brief sketch suggests that the various branches of mathematics should not be taught as distinct, and that the analogies existing should be pointed out even in elementary work. Also special importance should be attached to stressing the following notions :

(i) A set of elements of one system regarded as an element of a new system.
(ii) Congruence.
(iii) Correspondence and isomorphism.
(iv) Invariance under a correspondence.
(v) Duality.

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A. R. R.

1305. The imaginary right angle of Geometry will not build houses but the nearly perfect right angle which masons and carpenters use is responsible for many marvellous things.—Mahatma Gandhi, *The Illustrated Weekly of India*, Aug. 21, 1938. [Per Mr. T. S. Venkatraman.]

1306. MATHEMATICIAN

By Edward Weismiller

Running upon the morning, with no sound,
The hawk inscribes geometry in air ;
The curve he made ascending from the ground
Froze into space as it was plotted there.

Beneath the compass of his wings appear
Such fabulous designs as children make :
He wheels above the dark, untidy sphere
With arcs like sequins glittering in his wake.

All order is implicit in his flight ;
This is the sketch from which all wonder grew.
He soars : the world spins down and out of sight ;
And space is curved. But this he always knew.

—*Harper's Monthly Magazine*, October 1937, p. 479. [Per Mr. E. B. Escott.]

THE SOLUTION OF A CERTAIN CLASS OF DIFFERENTIAL EQUATIONS.

By M. C. K. Tweedie.

1. We will consider first the second order partial differential equation

$$\alpha\frac{\partial^2 y}{\partial x^2}+\beta\frac{\partial^2 y}{\partial x\,\partial t}+\gamma\frac{\partial^2 y}{\partial t^2}+\delta\frac{\partial y}{\partial x}+\epsilon\frac{\partial y}{\partial t}+\eta y=0, \quad \ldots\ldots\ldots\ldots(1)$$

where α, β, γ, δ, ϵ, η are all functions of x or t only.

Suppose the six coefficients are functions of x. Then since

$$\frac{\partial}{\partial t}\left(\alpha\frac{\partial^2 y}{\partial x^2}+\beta\frac{\partial^2 y}{\partial x\,\partial t}+\gamma\frac{\partial^2 y}{\partial t^2}+\delta\frac{\partial y}{\partial x}+\epsilon\frac{\partial y}{\partial t}+\eta y\right)=0,$$

we have

$$\alpha\frac{\partial^2}{\partial x^2}\left(\frac{\partial y}{\partial t}\right)+\beta\frac{\partial^2}{\partial x\,\partial t}\left(\frac{\partial y}{\partial t}\right)+\gamma\frac{\partial^2}{\partial t^2}\left(\frac{\partial y}{\partial t}\right)+\delta\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}\right)+\epsilon\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)+\eta\frac{\partial y}{\partial t}=0,$$

for none of the six coefficients contains t.

Therefore if $y=Y(x,t)$ is an integral of the given equation (1), then $\dfrac{\partial Y}{\partial t}$, and hence $\dfrac{\partial^n Y}{\partial t^n}$, where n is any integer, are integrals of the equation. Hence also the integral of Y with respect to t keeping x constant is a solution of the equation, and so too are the repeated integrals.

If the integral of Y with respect to t keeping x constant is denoted by $I(Y)$, and if $I^2(Y)=I(I(Y))$, and so on, then

$$y=\left(a_1 I+a_2 I^2+a_3 I^3+\ldots+b_0+b_1\frac{\partial}{\partial t}+b_2\frac{\partial^2}{\partial t^2}+\ldots\right)(Y) \quad \ldots\ldots(2)$$

is also a solution if the series are convergent.

Now $y=0$ is evidently an integral of equation (1) and therefore

$$y=(a_1 I+a_2 I^2+a_3 I^3+\ldots)(0)$$

is an integral.

Since $\dfrac{\partial I(0)}{\partial t}=0$ and $\dfrac{\partial I^2(0)}{\partial t}=I(0)$, etc.,

we have $I(0)=\phi_0(x)$, $I^2(0)=t\phi_0(x)+\phi_1(x)$,

$$I^3(0)=\frac{t^2}{2!}\phi_0(x)+t\phi_1(x)+\phi_2(x),$$

and so on, and therefore

$$y=\left(a_1+a_2t+\frac{a_3t^2}{2!}+\frac{a_4t^3}{3!}+\ldots\right)\phi_0(x)$$
$$+\left(a_2+a_3t+\frac{a_4t^2}{2!}+\ldots\right)\phi_1(x)$$
$$+(a_3+a_4t+\ldots)\,\phi_2(x)$$
$$+\ldots$$

is an integral of the equation (1). This solution may be written

$$y=\sum_{n=0}^{\infty}\phi_n(x)\frac{d^n}{dt^n}f(t), \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where $$f(t)=a_1+a_2t+\frac{a_3t^2}{3!}+\ldots+\frac{a_{n+1}t^n}{n!}+\ldots,$$

and is an arbitrary function because the a's are arbitrary. The ϕ's are found by substituting this expression for y in the given equation. By equating to zero the coefficients of the various derivatives of $f(t)$ in the equation so obtained, thus giving us a free choice of $f(t)$, we obtain the infinite set of ordinary differential equations

$$\left.\begin{aligned}
&\alpha\phi_0''+\delta\phi_0'+\eta\phi_0=0, && \text{giving } \phi_0(x);\\
&\alpha\phi_1''+\delta\phi_1'+\eta\phi_1=-(\beta\phi_0'+\epsilon\phi_0), && \text{giving } \phi_1(x);\\
&\alpha\phi_2''+\delta\phi_2'+\eta\phi_2=-(\beta\phi_1'+\epsilon\phi_1+\gamma\phi_0), && \text{giving } \phi_2(x);\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\alpha\phi_n''+\delta\phi_n'+\eta\phi_n=-(\beta\phi_{n-1}'+\epsilon\phi_{n-1}+\gamma\phi_{n-2}), && \text{giving } \phi_n(x);\\
&\qquad\text{etc.}
\end{aligned}\right\}\quad\ldots(4)$$

Since the original equation is of the second order, it can have only two independent solutions. It is consequently sufficient to introduce only two constants of integration in the solution of these equations for the ϕ's. The terms containing one constant may be taken with one arbitrary function of t, while those containing the other constant may go with another arbitrary function of t. The introduction of more than two constants gives a solution of the form of equation (2) in which Y is in this case a solution obtained by introducing only two constants. The form of the solution depends on where the two constants are introduced. It is often best to introduce one with ϕ_0 and the other with ϕ_1, but it should always be verified that the two constants multiply distinct functions.

A set of ordinary differential equations similar to (4) is obtained when the six coefficients are functions of t instead of x, in which case the substitution is

$$y=\sum_{n=0}^{\infty}\phi_n(t)\frac{d^n}{dx^n}f(x),$$

and again f is arbitrary.

The solution of the partial differential equation (1) is thus reduced to the successive solution of a series of ordinary differential equations. The method could evidently be used for equations of higher orders.

2. The solution of equation (1) expressed by (3) may also be derived in the following way from the solution of the equation

$$\alpha\frac{d^2y}{dx^2}+(\beta D+\delta)\frac{dy}{dx}+(\gamma D^2+\epsilon D+\eta)\,y=0 \quad \ldots\ldots\ldots\ldots\ldots(5)$$

in which D is a constant.

Substitute in (5) $y=\Sigma\phi_n(x)D^nC$, where C is a constant, and equate the coefficients of the various powers of D in the resulting equation to zero. This gives equations (4) for the ϕ's. The solution of the partial differential equation (1) can be deduced by substituting d/dt for D and $f(t)$ for C in $y=\Sigma\phi_n(x)D^nC$.

2·1. The equation (5) can sometimes be solved by the method of Frobenius, *i.e.* by substituting $y=x^\mu\Sigma a_nx^n$, but the D's then often occur in the result in positions where their interpretation as operators is difficult. The substitution of d/dt for D and $f(t)$ for C in the solution of (5) in the form $y=x^\mu\Sigma x^na_nC$, where C is an arbitrary constant, gives the solution of equation (1) immediately, however, in the special case in which a_n is a polynomial in D.* This case occurs when D does not appear in the indicial equation, *i.e.* when

$$\operatorname*{Lim}_{x\to0}\left(\frac{x\beta}{\alpha}\right)=0,\quad \operatorname*{Lim}_{x\to0}\left(\frac{x^2\gamma}{\alpha}\right)=0\quad\text{and}\quad \operatorname*{Lim}_{x\to0}\left(\frac{x^2\epsilon}{\alpha}\right)=0.$$

Example.

$$x^2\frac{\partial^2y}{\partial x^2}+\frac{\partial^2y}{\partial x\,\partial t}+\tfrac{2}{3}x^4\frac{\partial^2y}{\partial t^2}+2x\frac{\partial y}{\partial x}+2x^2\frac{\partial y}{\partial t}=0.$$

The coefficients are all functions of x and therefore substitution (3) is used. The differential equation for $\phi_n(x)$ is

$$x^2\phi_n''+2x\phi_n'=-(\phi_{n-1}'+2x^2\phi_{n-1}+\tfrac{2}{3}x^4\phi_{n-2}),$$

which can immediately be integrated to

$$x^2\phi_n'=-\int\{\phi_{n-1}'+2x^2\phi_{n-1}+\tfrac{2}{3}x^4\phi_{n-2}\}\,dx,$$

which again can be integrated.

$$\phi_0''+\frac{2}{x}\phi_0'=0\,;\ \text{so we may take}\ \phi_0=\frac{a_0}{x}+a_1.$$

$$-x^2\phi_1'=\int\left\{-\frac{a_0}{x^2}+2a_0x+2a_1x^2\right\}dx=\frac{a_0}{x}+a_0x^2+\tfrac{2}{3}a_1x^3,$$

giving
$$\phi_1=a_0(\tfrac{1}{2}x^{-2}-x)-\tfrac{1}{3}a_1x^2.$$

* I am indebted to Mr. R. L. Goodstein for pointing this out to me.

Proceeding thus, we get also

$$\phi_2 = a_0\{\tfrac{1}{9}x^3 + \tfrac{1}{6}x^{-3}\} + \tfrac{1}{3}a_1 x,$$

$$\phi_3 = a_0\left\{\frac{2x^5}{135} - \frac{x^2}{9} + \frac{1}{3x} + \frac{1}{24x^4} + \frac{\log x}{3x}\right\} + a_1\left\{\frac{x^6}{189} - \frac{x^3}{18} - \frac{\log x}{3}\right\},$$

etc.

The solution is therefore

$$y = f(t) - \frac{x^2}{3}\frac{df}{dt} + \frac{x}{3}\frac{d^2f}{dt^2} + \left\{\frac{x^6}{189} - \frac{x^3}{18} - \frac{\log x}{3}\right\}\frac{d^3f}{dt^3} + \dots$$
$$+ \frac{1}{x}F(t) - \left\{x - \frac{1}{2x^2}\right\}\frac{dF}{dt} + \left\{\frac{x^3}{9} + \frac{1}{6x^2}\right\}\frac{d^2F}{dt^2}$$
$$+ \left\{\frac{2x^5}{135} - \frac{x^2}{9} + \frac{1}{3x} + \frac{1}{24x^4} + \frac{\log x}{3x}\right\}\frac{d^3F}{dt^3} + \dots$$

where $f(t)$ and $F(t)$ are both arbitrary functions. The series are assumed convergent.

If this solution is taken as Y, then equation (2) gives other solutions.

3. The above method of solving partial differential equations suggests a method of solving ordinary differential equations.

The equation

$$\alpha\frac{d^2y}{dx^2} + \beta\frac{dy}{dx} + \gamma y + \delta = 0,$$

where the four coefficients are functions of x alone, can be solved by substituting

$$y = \sum_{n=1}^{\infty} \phi_n(x), \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(6)$$

which gives

$$0 = \delta + \alpha\phi_1'' + \alpha\phi_2'' + \alpha\phi_3'' + \dots$$
$$+ \beta\phi_1' + \beta\phi_2' + \beta\phi_3' + \dots$$
$$+ \gamma\phi_1 + \gamma\phi_2 + \gamma\phi_3 + \dots .$$

These terms can be grouped in an infinite variety of ways to give differential equations for the ϕ's so that the solution reduces to a series of ordinary integrations or to a series of differential equations that can be easily solved. For example

	or	or
$\phi_1'' = 0$	$\alpha\phi_1'' = -\delta$	$\beta\phi_1' = -\delta$
$\alpha\phi_2'' = -(\beta\phi_1' + \delta)$	$\alpha\phi_2'' = -(\beta\phi_1' + \gamma\phi_1)$	$\beta\phi_2' = -(\alpha\phi_1'' + \gamma\phi_1)$
$\alpha\phi_3'' = -(\beta\phi_2' + \gamma\phi_1)$	$\alpha\phi_3'' = -(\beta\phi_2' + \gamma\phi_2)$	$\beta\phi_3' = -(\alpha\phi_2'' + \gamma\phi_2)$
. . . . .	. . . . .	. . . . .
$\alpha\phi_n'' = -(\beta\phi_{n-1}' + \gamma\phi_{n-2})$	$\alpha\phi_n'' = -(\beta\phi_{n-1}' + \gamma\phi_{n-1})$	$\beta\phi_n' = -(\alpha\phi_{n-1}'' + \gamma\phi_{n-1})$

The best arrangement depends on the original equation. Some arrangements make solution a much more cumbersome process than others, but owing to the generality of the expression $\Sigma\phi_n(x)$ the method can succeed when the method of Frobenius and possibly other methods cannot. It is often best to introduce one constant of integration with ϕ_1 and the second with ϕ_2, and to arrange the equations so that the constants occur alternately in the ϕ's. More than two constants are unnecessary. The method can of course be applied to linear equations of any order, provided the coefficients contain the independent variable only.

For example, for the equation

$$\frac{dy}{dx}=\gamma(x)\,y, \text{ whose solution is of course } y=e^{\int\gamma\,dx},$$

taking $\phi_n'=\gamma\phi_{n-1}$, we have the solution

$$y=a\left\{1+\int\gamma\,dx+\int\gamma\int\gamma\,dx\,dx+\int\gamma\int\gamma\int\gamma\,dx\,dx\,dx+\ldots\right\},$$

which is the same as that given by the usual operational methods.*

M. C. K. TWEEDIE.

1307. A NOVEL MAGIC SQUARE.

SIR,—The remarks in "Notes By the Way" last week on mathematical calculations in connection with football recalled to me an intriguing problem bearing on cricket I managed to solve some few weeks ago.

On the impulse of the moment I started to make a "magic square" based on the total of 658 scored by England in last year's Test match at Trent Bridge.

In this excellent score there were four individual centuries—a record in Test cricket. These are all incorporated in the square, with as many as possible other notable scores made for England in Test matches, with a preference for those scored against Australia.

The problem was so to construct the square that the total of each horizontal, vertical and diagonal line, also each of the nine square blocks of four numbers shall equal the basic number 658.

216	102	214	126
98	242	100	218
115	203	113	227
229	111	231	87

The cricket scores were made as follow: E. Paynter 216, D. Compton 102, C. J. Barnett 126, L. Hutton 100. These four centuries were all made in the Trent Bridge Test. H. Larwood, 98 at Sydney, 1933; H. Sutcliffe, 115 at Sydney, 1924; M. Leyland, 111 at Melbourne, 1937; W. R. Hammond, 113 at Leeds, 1930; W. R. Hammond, 231 at Sydney, 1936; W. R. Hammond 227 at Christchurch, v. New Zealand, 1933.

W. Hammond's 227 is the only one not made in a Test against Australia.

—I am, sir, &c.,

A. GARDNER.

—*Nottingham Guardian*, May 13, 1939. [Per Mr. V. Randall.]

* H. Jeffreys, *Operational Methods*, Cambridge Mathematical Tracts, No. 23 (1927).

SOME DOUBTS ABOUT THE TEACHING OF MATHEMATICS IN SECONDARY SCHOOLS.*

By A. Blackwell.

"Please Sir, what's the use of learning this?"

I wonder how many times we have all—that is, if we are at all "approachable"—been asked this or similar questions? To-night I hope to show why the question is ever asked and, in my opinion, how to stop it arising in the minds of the pupils.

But first let me make clear this one fact. Although this is being presented to you in the month of February it was commissioned by your committee in December last, and though in the interim the Spens Report has been published it has not been used in any way in the preparation of this paper.

It is proposed, first of all, to deal briefly with the general theory of education and then see how far this is applied at the present day to the teaching of mathematics in particular, and finally, if change be needed, how best to accomplish it.

Now no theory of education can be taken as absolute and final, and what follows must of course be taken only as a personal opinion. Nevertheless it is the result of considerable reading, thought and discussion, and, I believe, represents the general concensus of opinion of, if I may say so, enlightened, present-day, English, democratic thought.

The Code of Regulations of the Board of Education opens with the words, "The purpose of the School is to form and strengthen the character and to develop the intelligence of the children entrusted to it and to assist the boys and girls to fit themselves for the work of life". More briefly, Sir Richard Gregory defines education as "the deliberate adjustment of a growing human organism to its environment". But it is very difficult to convey our thoughts on the subject in a single, brief sentence. However, it is unanimously agreed that in all our deliberations we must have in mind the pupils themselves, *with their interests*, varying as they do according to age and development. We must think as much about what *they like* to learn as about what *we* wish to teach them. It is of the utmost importance that children should enjoy their work. If a subject *is* worth while including in a curriculum, the pupils must not only learn that subject but learn to love it. In the past insufficient attention was often paid to the sentiments and feelings which may grow up about a subject, yet these may form the very foundations of permanent interest and should lead to a real enrichment of life. This necessity for the pupils being interested in a subject makes itself evident, not only from the positive effect just mentioned, but lack of it undoubtedly leads to definite adverse sentiments and distaste which will persist, if not throughout life, at any rate for a large number of years.

* A paper read to the Sheffield and District Branch of the Mathematical Association on 7th February, 1939.

Now it is agreed that these interests gradually change as the child advances in years and investigations have been carried out to determine what they are at each stage of development. It is found that young children are mostly interested when the facts presented are in themselves wonderful, startling or extraordinary. Later they pass from this collecting stage to a utility stage, when they wish to study topics that are important in the world of practical affairs. They are not interested in abstractions and generalisations except in so far as these are needed to explain their own experience, and they like to feel that the principles they study are actually useful, if not to themselves at the present moment, at any rate to somebody somewhere. They are interested in the practical activities of life as presented to them in the household, shops, streets and workshops ; in the activities and occupations of the community at large. The young desire to enter into this life with the full activity of their senses, hands and thought, to gain added experience of it, to hear about it and to share in it, or, where they cannot actually share in it, to imitate it. It is only at the age of fifteen or sixteen at the very earliest that the vast majority of children even begin to take an interest in academic work or learning for its own sake. These are advanced and very specialised interests appropriate to intellectual maturity, and teachers would do well not to *over*-stress logical and formal connections in the early stages. In the past too much reliance has undoubtedly been laid on the intrinsic logical interest of the various subjects, but, unfortunately, the aesthetic charm of formal demonstration is apparent mainly to adult minds only. This in no way implies that the teaching should be vague and impressionistic, for it is important that the child should possess a growing nucleus of exact knowledge which he thoroughly understands and can use with ease, whilst the teaching itself should stimulate him to real effort. Only thus will our pupils acquire habits of hard work and intellectual honesty.

According to the Science Masters' Association,* analysis of views generally held shows that the numerous reasons urged to justify the inclusion of any subject in the curriculum can be roughly classified under three headings, namely :

(α) utilitarian or vocational—it helps the pupils in their everyday life or may be necessary in their future occupations ;

(β) cultural—its inclusion is desirable because it forms an essential part of our social heritage ;

(γ) disciplinarian—it teaches them to think : it sharpens their minds.

I shall take it for granted that there are few here to-night who would take the *purely* utilitarian or vocational aspect as the *sole* reason for the inclusion of any subject in the curriculum of the

* *Interim Report of the Sub-committee* (1935) *of The Science Masters' Association on The Teaching of General Science*, p. 10.

ordinary secondary school as we know it, at any rate for children up to the age of sixteen. True it may give added stimulus to the pupil to acquire a large store of such knowledge, but the diversity of subjects for which it would be necessary to give facilities would make it a practical impossibility. And we would like to make it clear that while the *pre*-preparation of experts and technicians in all subjects is part of the function of the schools, it is not the only function ; further, the fact that I have stressed the importance of arousing the interest of the pupils up to the age of fifteen or sixteen by the practical and occupational aspect of any subject belongs rather to the second classification—the cultural value of the subject. This I do not propose to elaborate. Surely everything we teach in school should illuminate as far as possible the actions, thoughts and feelings of this civilisation in particular and the whole world in general, or bring out the effects of previous ages on the present.

The third claim advanced in favour of the teaching of various subjects in schools, namely the disciplinarian, rests on belief in the possibility of " transfer of training ". Firm belief in this is still held by many outside, and indeed, I believe, by some, at least, inside the profession. It assumes that mental powers (*e.g.* memory, observation, reasoning power, imagination, etc.) however gained would be applicable to any department of human activity. This belief was apparently still extant in 1918, for we find a Government Report speaking of science that " it should train the mind of the student to reason about things which he has observed for himself and develop his powers of weighing and interpreting evidence ", almost, one feels, as if the mental attributes so obtained through science will have a similar carry-over into *any* activity. By 1924, however, official statements are to the effect that the subject is " debatable ", and in 1930 the British Association Report on Formal Training is able to state that " experimental evidence has shown quite definitely that the possibilities of transfer of training are much smaller at any rate than had been formerly supposed ".

So much for the grounds on which our choice of subjects or topics to be taught should rest.

Closely connected with this is the question, " Where should we *start* the teaching of any individual topic? " The answer comes immediately—" by consideration of some aspect of our topic where it touches daily life—of the pupil himself or of some member of the community at large ". This often means taking a particular case first before arriving at a general conclusion and may thus appear illogical. However this need not worry us, as up to the age of fifteen or sixteen the child is not interested in logic. And it must be noted that usually this is the historical order in which general principles were arrived at originally. In fact, to follow the historical sequence is now considered the best in certain subjects.

From these starting-points within the child's actual experience, or the activities of some other member of the community in which

he is interested, we should endeavour to extend his knowledge and experience to approximate to that of the life and environment (in the widest sense) into which he will go on leaving school. This of course means that a *complete, wide view* of the subject must be given, showing how it affects life at as many points as possible. This is of paramount importance. Thus if a person were studying, say, the geography of Europe, it would be preposterous to spend all the available time on the geography of England alone. There are at least two reasons for not doing so. Firstly, England is very different from most of the rest of Europe and so cannot be taken as representative, and, secondly, we should not even have a complete view even of England if we failed to study its interactions and relationships with the remainder of the Continent. "What would they know of England who only England knew?"—let alone Europe, which they were supposed to be studying.

Let us now examine the present position of the teaching of mathematics up to School Certificate standard, and ask ourselves frankly what ends have we in view? what are the results? and how far does it fit in with the general theory outlined above? Equally important, we must answer these questions honestly.

First, then, what ends have we in view? I suggest the present syllabus is framed to produce what we might call "professional mathematicians", at any rate it is laid down with a view to the requirements of future VI Form mathematical specialists who will, it is hoped, eventually go on to a mathematics degree at the university. As to whether this *is* the best foundation for those future specialists, I will leave for the moment, and will now consider the justification of such a course. Perhaps it could be justified in the days when there were few institutions of what we call secondary or grammar school standard, where a comparatively large proportion of the pupils did go on to the university. But in the last twenty or thirty years there has been a very big increase in the number of secondary schools and a tremendous increase in the number of pupils attending them, without a corresponding increase in the number of pupils going on to the universities (less than 3 per cent. proceed to an honours degree of *any* kind). This means that for the large majority of our pupils we have been giving them an education not suited to their wants and needs. In a rather snobbish way we have allowed them to listen to the marvellous stuff (supposedly) required by the future mathematical specialists and not bothered at all whether we were fostering a like or a dislike for the subject amongst the vast majority of the pupils. Of course we have bothered *about* them—and bothered them in doing so. We have all harassed and bullied them into working hard at the subject in order to get a reasonable number through the School Certificate Examination.

No doubt a lot of people are still convinced that the present course is justified on the grounds that no matter what happens to

the general mob of citizens we *must* have highly trained specialists in all fields of human activity. I agree that we do need these specialists, but submit they would still be produced *and probably of a still better quality*—and thus be still more useful to the community—if the foundations of their study had been really broad and comprehensive. The ground that has thus once been turned over can be tilled so much more easily and deeply when they reach the VI Form and the university.*

Undoubtedly another important aim has been to train the children's minds in logical and clear thinking, and this is all to the good and should always be at the back of the teacher's mind.

But let us now get back to the second question, viz. what is the result of the present aims and methods in the teaching of mathematics?

I say it is appalling, and anyone who says otherwise is deluding himself and not facing the facts squarely and honestly. I submit that at the present moment there is a horror of mathematics among the general public and active opposition to it. As I said at the beginning, " How many of us have been asked over and over again, ' what's the good of learning this? ' " That fact in itself shows at any rate that there is no positive love for it amongst those individuals. If there were, those pupils would never stop to think about the question. Does the normal, healthy child stop in the middle of a game of football or hockey to ask the instructor " What's the good of doing this? " No. They love it far too much, their minds are too much occupied for the question to have time to be formed. And, be it noted, we do not let them play games just because they *like* to, but because we believe they will get considerable benefit from doing so. However, I do not suggest we turn our schools into miniature physical training colleges. Nevertheless, without identifying myself in any way with the political ideology which first culled the phrase, I think there is a lot to be said for " Kraft durch Freude " throughout the whole realm of education.

Let us think for a few moments about this plethora of so-called " popular " scientific books. What does " popular " mean? Simply this—it is non-mathematical. That is the one common factor in all the advertisements for all of them. And why this great necessity for the books to be non-mathematical? Simply that the authors and publishers know from bitter experience that the public in general will not buy books which are mathematical, which is tantamount to saying that the general public—even the highly intelligent but non-mathematical members—have a positive distaste for it.

* Since writing this I have found an endorsement of the idea in the Preface of Professor F. L. Griffin's book *An Introduction to Mathematical Analysis*, a book which covers a very wide range of mathematical topics, in which he says " Specialist students, as experience has shown, acquire an excellent command of mathematical tools by first getting a bird's-eye view of the field, and then proceeding to perfect their technique ". Surely if this is true of college students, it is even more necessary that we should give such a view to pre-school-certificate pupils.

If further evidence were required I would draw your attention to a certain gentleman, Lancelot Hogben, who wrote a book *Mathematics for the Million*—written, whilst the author was in hospital during a long illness, for his own fun. " A few friends from among *the million or so intelligent people who have been frightened by mathematics while at school* (I am quoting his own words) persuaded him to publish it." He goes on, " *It will have fulfilled its aim if it stimulates the interest and removes the inferiority complex of some of the million or so who have given up hope of learning through the usual channels.*" I almost blush to have to read such a castigation of my own profession. I would not worry so much if I took it only at its face value—the writings of a sick man in hospital (I am afraid he does not say what type of illness it was)—but *the book has become a best seller* in both England and America. Apparently there was a crying need for it.

Westaway says, " Though VI Form specialists are doing solid work (of a very restricted type, it must be added) the amount of mathematical work being done by all the other pupils who have obtained the School Certificate is, as a rule, slight, too slight and much too academic for the fostering of a life-long interest in the subject." And that, be it noted, is about those who have already obtained School Certificate—what about those who have not even reached School Certificate, when most of them will leave school altogether?

As a result of all this positive distaste for the subject, what has become of all the formal discipline of the mind? I admit it would be difficult to measure—and chiefly, I suggest, because of its ultra-microscopic size.

However, I think I have said enough to show that even though our teaching of mathematics up to School Certificate standard may be successful in getting the requisite number through that examination and producing a sufficient number of so-called specialists, the general effect on the vast majority of our pupils is unsatisfactory and tends to evoke anything but a lively interest in the subject. I now propose to find out where it is wrong by testing it with regard to the criteria laid down in the general theory of education with which I began this talk. Let it be quite clear that I am going to test *school* mathematics as it has to be taught now in order to fulfil the requirements of the School Certificate Examination. We will subsequently think about mathematics without the adjective " school ".

First, does it interest the pupils? This means, is it bound up with the lives of the pupils themselves in what they do and see in their homes, in the streets and shops, and what they hear about in the lives of other members of the community such as builders, carpenters, engineers of all kinds, architects, surveyors and navigators? For we must remember that unless it is, it will not interest them. Of course, the answer comes pat, " Yes. We all go to the grocer's, sell pigs and buy coal, find the height of cliffs and the length of a

guy rope on a ship's mast". True, a miserable bit of arithmetic, a little trigonometry and Pythagoras, including far too many abstruse, inverse and complex problems on stocks and shares such as rarely occur in real life. Admitted, too, that modern textbooks give formulae taken from real life for exercise in logarithms and change of subject of a formula, but the latter is not needed anything like as much in actual life as we are led to believe.

But what of the major portion of algebra and geometry? I have never yet heard of a farmer who forgot the price of sheep and cows and had to work backwards from two bills he had. And if anyone can give me a really practical example of the use of the angle between the tangent and the chord being equal to the angle in the alternate segment, I should be very pleased, or even an example of the use of the sum of the angles of a triangle being equal to two right angles.

I feel sure that here we are getting at one of the real roots of the trouble; there is no real interest, the subject's dead. There may be a few odd points where we can show a practical use—we may be able to find more if we do a bit of research—but taken as a whole, at present, school mathematics touches life at far too few points to make any but the very few favoured ones have a lively interest in it. The mathematics we teach has not enough starting-points in the child's experience, and the ends to which we lead them certainly do not reflect the life of the community at large in anything like a justifiable degree.

Under the three types of argument I mentioned into which the justifications for teaching a subject can be placed, viz. utilitarian, cultural and disciplinarian, I think that the latter is the only one that can be seriously advanced in favour of present-day mathematics. Of course, every one agrees mathematics is a good mental discipline, fostering logical reasoning and so forth, but let me reiterate that such mind training has been shown conclusively by modern research to have very little "carry-over", as it is sometimes called, into other human activities. By all means let us try to nourish it, but let not this be the only justification for the inclusion of our subject in our curriculum. Let us take a warning from another subject that had only this one reason—Latin. We are certainly witnessing the "death agonies" of Latin—*as it has been taught*, at any rate. They are trying to shift their weight off this foot of formal discipline on to the cultural foot, but personally I would scrap the subject from School Certificate altogether.

And with regard to the cultural and utilitarian values of our present-day mathematics, I feel these are really so small that we are in danger of being in the position of Latin if we are not careful. Except for some of the arithmetic, most of which was learnt at the elementary school, I do not think we can say that much of our mathematics either assists anybody to get their daily bread or illumines very much of the environment in which they live or will live on leaving school. So what? Are we going to be in the same

position as Latin? What do the children get? Plenty of formal discipline, a tiny little bit of cultural value and about the same amount of utilitarian value—one outside ace and two very small trumps! Truly a raw deal.

Well, can we have a New Deal?—or, being mathematicians, we would probably prefer a Square Deal. My answer is, "Yes, definitely yes". Mathematics can still retain that outside ace of mental discipline *and* the ace, king of trumps, cultural and utilitarian values. First and foremost, can we develop interest in the subject? which means, for children up to the age of fifteen or sixteen, does mathematics touch life at many points? Let us hear what Westaway thinks. "Mathematics touches life at so many points that the all-round training of a mathematician is a very long business. Consider the mathematics of physics alone: what a vast field! The field is, of course, ploughed up and sown by the teachers of physics, with the result that there is always a rich harvest for the mathematical staff to reap." He adds (significantly), and in italics, "Is that reaping always done?" Elsewhere he talks of "the mathematics of physics, of chemistry, of engineering, of machinery and structures, of aviation, of astronomy, of biology, of statistics". Hogben also seems to think that mathematics and life *are* rather closely allied, when he calls his first chapter of *Mathematics for the Million* "Mathematics—the Mirror of Civilisation", and actually criticising present-day teaching says, "As mathematics is now being taught and expounded in our schools no effort is made to show its social heritage, its significance in our social lives, the immense dependence of civilised mankind upon it. . . . The first men who dwelt in cities were *talking* animals. The man of the machine age is a *calculating* animal. We live in a welter of figures—cooking recipes, railway time tables, unemployment aggregates, fines, taxes, war debts, overtime schedules, speed limits, bowling averages, betting odds, billiard scores, calories, babies' weights, clinical temperatures, hours of sunshine, motoring records, power indices, gas-meter readings, bank rates, freight rates, death rates, discount, interest, lotteries, wavelengths and tyre pressures . . . ratio, limits, acceleration are not remote abstractions, dimly apprehended by the solitary genius, they are photographed upon every page of our existence". The Board of Education in their "Suggestions for Teachers" mention mensuration, surveying, mechanics and manual instruction, the use of the vernier, micrometer screw gauge, volumes and densities, U-tube work, pulleys, levers, pendulum and Hooke's Law for quite young children. We thus see that mathematics does touch life at many points indeed and could therefore arouse interest in young adolescents if we chose the topics properly from a wide range.

And besides arousing this essential interest, choice of correct subject-matter will also bring *at the same time* its utilitarian and, to a large extent, its cultural value. With regard to the former I do not wish it to be felt that I want to make plumbers, or carpenters, or architects, or engineers of all the pupils, but I do want

him to feel if he does become one of these (or many other professions) and runs up against the mathematical aspect of the topic that it is something within his capabilities. Or at any rate, I do not want him to feel a repugnance for the mathematical aspect of the subject. I want him to feel the reverse. It was for such people as these that Hogben wrote *Mathematics for the Million*. And whilst we are on the subject of utility, let me point out that it is going to be much better for even the specialist if he has dealt with a wide range of mathematical topics, not only that the ground has once been turned over and will therefore be so much more easily tilled later, but a specialist in one branch of work frequently has to appeal to other specialists in other branches, and therefore it will be much better for them both if they have some small experience of each other's work, at least in the earlier stages of their education.

As regards formal discipline, I think I can leave that safely in the hands of any of our present-day mathematics teachers.

The question now arises as to what curriculum we shall require and what methods and technique adopt. Here I expect no two of us would agree in every minute detail—I do not think it essential or desirable that we should. I will therefore consider only broad issues.

To start with, the curriculum must be complete in itself. There must be no "loose ends", whose significance is only fully appreciated by further work in the VI Form. We must keep well to the forefront of our minds that vast majority of our pupils who are *not* going to be mathematical specialists. If we do this, as I have already pointed out, the specialists will be catered for *ipso facto* and *a fortiori*. This means a drastic cutting down of purely manipulative exercises in algebra and geometry. Take for example the famous chapter in all algebra textbooks and a dead cert question for School Certificate, viz. surds, and especially rationalising denominators. True we want some. I frequently want sin 45° and sec 30°, so by all means teach how to find such simple things as $1/\sqrt{2}$ and $2/\sqrt{3}$ easily and without having to get a set of tables. But whatever value is there in

$$\frac{(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{3}+\sqrt{5})}{(\sqrt{2}+\sqrt{3}-\sqrt{5})(\sqrt{3}-\sqrt{2}+\sqrt{5})}$$

or any such ridiculous quantity? When is it needed? Can anyone give me a practical example of it, even in advanced work? Yet that question is taken from a much-used book recently published and designed specifically for School Certificate *elementary* mathematics (the italics are the author's—not mine!)—not the so-called Additional Mathematics. Of course, we shall still require a lot of drill work, but I suggest it could be cut down very considerably; not, admitted, and continue to get our usual number through the School Certificate Exam., where we have to cover every degree of theoretical complexity because of the supposed needs of the future specialists.

As I see it there are three types of exercise required.

(α) Those in connection with some new idea or principle where the numbers chosen are very simple so that the whole attention is focused on the principle and there is no need to worry about the arithmetic.

(β) Exercises to give skill in pure manipulation and drive home the principle by sheer repetition.

(γ) Problems taken from real life where the principle adduced under (α) and requiring some degree of (β) is required.

But at present the examinations require chiefly highly complex examples of the purely manipulative (β) type and the few of the problem (γ) type which are set are usually hypothetical and still require a very high degree of manipulative power; I suggest this is a pure waste of time and human effort. There are three types of pupils, (1) a few who have the so-called mathematical type of brain and so can very soon do pretty well any example you give them, (2) the few who are anything but mathematical and who cannot be guaranteed to get the 1000th example correct, and (3) the intermediate mass. By giving a large number of manipulative exercises to all these pupils we do very little good to most of them. The good ones do not need them, the very poor ones will never improve, but we go on hammering away in the hope that one or two of those who are on the border line of a failure may just remember how to do the particularly nasty example which will come in the examination. As an illustration, let me return to the example of rationalising denominators. In the textbook just quoted there are thirty-four examples on this topic in the actual chapter besides heaps more in countless "Tests in Manipulation" and "Revision Tests" at the end of the book. And these thirty-four do not include the necessary preliminary exercises in the multiplication of two irrational binomials. All told, there are 110 examples in that section; I could not be bothered to count those in the various tests at the back of the book. I will admit that everybody was not expected to do them all. Some were starred, showing that only the best would be able to tackle them. Do you know how many were starred? *Three*—out of 110. There were six examples which might be referred to as being of the problem type rather than the purely manipulative. They ranged from, "if $y=(z+1)/(z-1)$ and $z=x^2-1$, find the value of y when $x=1+\sqrt{7}$", to, "find the value of x to 3 figures if $x+x\sqrt{3}=10$". And then we wonder why the interest in mathematics is killed.

Westaway says, "The regular working of exercises is essential for the illuminating of ideas and impressing them on the mind. Nevertheless, the working of exercises tends to dominate our work far too much and to consume time that might far more profitably be devoted to the tilling of now neglected ground of great interest."

From all this I think you will see that when I ask for the basis to be broadened, and that what *is* done, to be done not so deeply

as at present, you will realise it is quite possible to do so rate with algebra. The same is true of geometry. "Mathematics books", Hogben says, "are largely packed with exercises which are designed to give proficiency in workmanship. This makes us discouraged, because of the immense territory which we have to traverse before we can get insight into the kind of mathematics which is used in modern science and social statistics. The fact is that modern mathematics does not borrow so very much from antiquity. To be sure, every useful development in mathematics rests on the historical foundation of some earlier branch. At the same time every new branch liquidates the usefulness of clumsier tools which preceded it. Although algebra, trigonometry, the use of graphs, the calculus all depend on the rules of Greek geometry, scarcely more than a dozen from the two hundred propositions of Euclid's elements are essential to help us in understanding how to use them. The remainder are complicated ways of doing things which can be done much more simply when we know later branches of mathematics. For the mathematical technician these complications may prove a useful discipline. But the person who wants to understand the place of mathematics in modern civilisation is merely distracted and disheartened by them."

Now we all agree we want to have mental discipline and give the pupils some insight into mathematical rigour. But we must remember we must not force it down their throats at too early an age, or bang will go the interest and *along with it the formal discipline* we are trying to give them. I therefore suggest that *during the last year* we do give them a fairly *deep* study of one branch of mathematics and bring out the intrinsic logical aspect. As to which branch we choose, it does not matter much, but I suppose most of us will be thinking of formal geometry. In fact, when you come to think of it, geometry, *even at present*, is about the only bit which is developed at all logically! Both arithmetic and algebra are more or less hotch-potches of various odds and ends.

I can imagine some one saying, "Yes, and what about my poor future specialists?" I suggest they will be quite all right. You have still two whole years before Higher Certificate and very few of them will go up to the university before a further year has elapsed. And even in the VI Form everything is not done with full mathematical rigour, at first at any rate. Newton himself did not discuss the calculus with complete rigour—that has only been done by modern analysis.

So we arrive at the conclusion that we should arrange our courses in as liberal a manner as possible, and that they should be designed to cover a wide range, at any rate in the early stages. That this can be done I have shown, if we do not treat the topics so deeply as at present and cut out much of this really unnecessary manipulation.

Now a certain body of teachers have already tackled their subject in such a manner and have got it working. I refer to the science

men who have now produced the subject "General Science". They seem to have been thinking on these lines since about 1915, and I think we could benefit by their experience and thought in deciding how the curriculum can be framed from such a welter of applications of mathematics which undoubtedly exist. I will therefore read to you in their own words the methods they adopted. The only change necessary will be to read the word "mathematics" wherever they have the word "science".

"(A) Three men set to work to make a list of points at which pupils made close contact with the scientific aspect of their environment and from this list they derived a collection of teaching material necessary to bring the pupil into an intelligent relation with that environment. In other words, they tried to find out the things a boy wants to know, the questions a boy wants to be answered.

"(B) A syllabus derived from such a list of 'contacts', while it would probably possess many advantages, would not necessarily possess all the required qualities. It might tend to consist of a collection of isolated fragments with no intelligible relation between them. To give pupils a superficial acquaintance with a miscellaneous assortment of facts is not to teach Science. What gives unity to any branch of Science is the appreciation of the common factors, and the common factors are the underlying scientific principles and laws. To give coherence to our syllabus, three other men, therefore, selected a few very important fundamental scientific ideas, and worked outwards from these to a second list of subject-matter, which exemplified them.

"When these two lists, List A and List B, were put side by side, it was found that there was a considerable amount of common material. This was called List AB and it formed the backbone of our syllabus. It is clear that every item in it justifies its presence *inter alia* on two grounds: (α) it helps to give the boy an intelligent understanding of his environment, and (β) it brings him into sympathetic contact with the underlying scientific principles.

"Apart from List AB, there is very little in our syllabus except such additional material as had to be included on the ground that it was a necessary preliminary to the comprehension of AB." *

As six people are mentioned as being necessary to work out the full details of such a syllabus, this has not yet been done, but I do feel that besides arithmetic, trigonometry, algebra and plane geometry, we shall also need solid geometry, conics, projections, calculus, statistics and applied mathematics. In fact, the last named, taken in its widest sense, probably covers the whole field. A. B.

1308. La France est peut-être le seul pays au monde où 2+2=toujours 4. Mais à Cambridge 2+2=l'infini, et à Oxford personne ne s'abaisse jusqu'à faire une addition.—André Maurois, *Les Anglais.* [Per Mr. F. J. Wood.]

* *The Teaching of General Science* (Science Masters' Association), p. 19.

THE LAPLACE TRANSFORMATION METHOD IN ELEMENTARY CIRCUIT THEORY.

By J. C. Jaeger.

1. Heaviside's operational method has for some time been in fairly general use among engineers for solving problems in circuit theory, and appears in most of their textbooks. The average man who has read these seems able to manipulate the "expansion theorem" and to solve problems in which E.M.F. is applied to an initially "dead" circuit, but he is not at all clear about the foundations or correctness of the theory and apt to be worried by any slightly more complicated problem, such as cases in which the circuit is not initially dead or problems involving switching operations (which are usually dealt with in the textbooks by artifice). It is now pretty well recognised that the Laplace transformation provides a satisfactory approach to the operational methods, but it is not perhaps fully realised just how simple it is and how extremely easy to teach.

The object of this note is to illustrate the applications of the method to easy problems in circuit theory. It is based on Professor Carslaw's article * in a recent *Gazette*, but the essential theory is covered in §§ 2 and 3 (from a slightly different standpoint †) in order to make the whole self-contained.

2. If
$$\bar{x}(p) = \int_0^\infty e^{-pt} x(t)\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$\bar{x}(p)$ is called the Laplace transform ‡ of $x(t)$; p is a real positive number supposed large enough to make the integral (1) convergent.§ This "bar" notation will be used throughout.

The apparatus of the method consists of a collection of Laplace transforms; those arising in ordinary problems are collected in the Table below; (2) to (7) are elementary integrals and (8) and (9)

* *Math. Gazette*, 22 (1938), 264.

† The method in precisely the form given here dates back at least to Bateman, *Proc. Cambridge Phil. Soc.*, 15 (1910), 423.

‡ Van der Pol, *Phil. Mag.* (7), 7 (1923), 1153, and in many subsequent papers has used in place of (1)
$$f(p) = p\int_0^\infty e^{-pt} h(t)\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1')$$
and he has been followed by many writers. The notation (1′) has the advantage that $f(p)$ corresponds exactly to the Heaviside operator for the function $h(t)$, whereas the $\bar{x}(p)$ of (1) would differ from it by a factor p. But the factor p of (1′) is simply carried as a passenger through the algebra and sometimes leads to increased complication, *e.g.* in replacing the simple idea of expansion in partial fractions by the use of the "Expansion Theorem". Also the form (1) (though not the notation) is used by Doetsch in his standard work, *Theorie und Anwendung der Laplace-Transformation* (Berlin, 1937), and by writers on the mathematical theory.

§ For example, if $x(t) = e^{+at}$, $\bar{x}(p) = \int_0^\infty e^{-pt} \cdot e^{+at}\, dt = \dfrac{1}{p-a}$, $p > a$.

are obtained most easily from (5) and (4) by differentiation with respect to the parameter a.

TABLE.

$\bar{x}(p)$	$x(t)$	
p^{-n}	$t^{n-1}/(n-1)!$, n a positive integer	(2)
$\dfrac{1}{(p+a)}$	e^{-at}	(3)
$\dfrac{a}{p^2+a^2}$	$\sin at$	(4)
$\dfrac{p}{p^2+a^2}$	$\cos at$	(5)
$\dfrac{a}{p^2-a^2}$	$\sinh at$	(6)
$\dfrac{p}{p^2-a^2}$	$\cosh at$	(7)
$\dfrac{p}{(p^2+a^2)^2}$	$\dfrac{t}{2a}\sin at$	(8)
$\dfrac{a^2}{(p^2+a^2)^2}$	$\dfrac{1}{2a}(\sin at - at\cos at)$	(9)

An important extension is contained in the following result :

THEOREM. *If $\bar{x}(p)$ is the Laplace transform of $x(t)$, then $\bar{x}(p+a)$ is the Laplace transform of $e^{-at}x(t)$.*

This follows immediately since

$$\int_0^\infty e^{-pt}[e^{-at}x(t)]\,dt = \int_0^\infty e^{-(p+a)t}\,x(t)\,dt = \bar{x}(p+a).$$

With the help of this theorem a large number of transforms which arise in practice may be derived from the results of the Table ;

e.g. if $\quad \bar{x}(p) = 1/(p+b)^n, \quad x(t) = e^{-bt}t^{n-1}/(n-1)!,$

if $\quad \bar{x}(p) = \dfrac{a}{(p+b)^2+a^2}, \quad x(t) = e^{-bt}\sin at. \quad \ldots\ldots\ldots\ldots(10)$

3. The method of solving an ordinary linear differential equation with initial conditions is as follows. Suppose we have to solve the second order * equation

$$(D^2 + a_1 D + a_2)x = F(t), \quad t>0, \quad \ldots\ldots\ldots\ldots(11)$$

* The equation of order n and simultaneous equations are dealt with in the same way, cf. Carslaw, *loc. cit.*

where D is written for d/dt, and x and Dx are to take the values x_0 and x_1 when $t=0$.

Multiply (11) by e^{-pt} (where p is supposed positive and sufficiently large to make all the integrals converge) and integrate with respect to t from 0 to ∞, then using the results

$$\int_0^\infty e^{-pt}\,Dx\,dt=\Big[e^{-pt}\,x\Big]_0^\infty+p\int_0^\infty e^{-pt}\,x\,dt=p\bar{x}-x_0,$$

$$\int_0^\infty e^{-pt}\,D^2x\,dt=\Big[e^{-pt}\,Dx\Big]_0^\infty+p\int_0^\infty e^{-pt}\,Dx\,dt=p^2\,\bar{x}-px_0-x_1,$$

we obtain from (11) the " subsidiary equation "

$$(p^2+a_1\,p+a_2)\bar{x}=\bar{F}(p)+px_0+x_1, \quad \text{.....(12)}$$

or

$$\bar{x}(p)=\frac{\bar{F}(p)+px_0+x_1}{p^2+a_1\,p+a_2}. \quad \text{.....(13)}$$

We have now determined the Laplace transform $\bar{x}(p)$ of the solution $x(t)$ of the problem, and have to find $x(t)$ from it. It is known * that if two continuous functions have the same Laplace transform they are identically equal. Thus if by any means we can find a function $x(t)$ whose Laplace transform is the right-hand side of (13) this will be the solution of our problem. For the simple values of $F(t)$ which arise in most problems $\bar{F}(p)$ is a rational function of p, and $\bar{x}(p)$ takes the form $f(p)/g(p)$, where $f(p)$ and $g(p)$ are polynomials in p, the degree of $f(p)$ being at least one lower than that of $g(p)$. Thus we can put $\bar{x}(p)$ in partial fractions which will express it as a sum of terms such as appear in the left column of the Table; the function $x(t)$ can then be read off from the Table.

For example, to solve

$$(D^2+1)x=1, \quad t>0$$

with $x=0$, $Dx=1$, when $t=0$.

Using (2) the subsidiary equation (12) becomes

$$(p^2+1)\bar{x}=\frac{1}{p}+1.$$

Thus

$$\bar{x}=\frac{p+1}{p(p^2+1)}=\frac{1}{p}-\frac{p-1}{p^2+1}.$$

* This follows immediately from Lerch's Theorem, Doetsch, *loc. cit.*, p. 35. A simple case of Lerch's Theorem which is adequate for our purpose is that if $\int_0^\infty e^{-pt}x(t)\,dt=0$ for every p greater than some fixed number, and if $x(t)$ is continuous, then $x(t)\equiv 0$. If two continuous functions $x_1(t)$ and $x_2(t)$ both have $\bar{x}(p)$ for Laplace transform we should have $\int_0^\infty e^{-pt}[x_1(t)-x_2(t)]\,dt=0$, and thus, by the result above, $x_1(t)\equiv x_2(t)$.

Therefore from (2), (4), and (5)

$$x = 1 - \cos t + \sin t.$$

The only mathematical technique required is that for expressing $\bar{x}(p)$ in partial fractions. If, as above, $\bar{x}(p) = f(p)/g(p)$, and if the roots of $g(p)$ are all different, say $\alpha_1, \alpha_2, \ldots, \alpha_n$, this may be done by using the result

$$\frac{f(p)}{g(p)} = \sum_{s=1}^{n} \frac{f(\alpha_s)}{(p-\alpha_s)g'(\alpha_s)}. \quad \ldots\ldots\ldots\ldots(14)$$

If some of the roots of $g(p)$ are repeated, or if it is desired to express $\bar{x}(p)$ in partial fractions with quadratic denominators, it is best to use the ordinary algebraic methods. Equation (14) corresponds to the Heaviside "Expansion Theorem", but the latter has an extra term arising from the factor p in (1′).

4. *E.M.F. $V(t)$ applied at $t=0$ to a circuit consisting of inductance, resistance, and capacity in series.*

I, V, Q will be used for current, E.M.F., and charge respectively, $\bar{I}$, $\bar{V}$, $\bar{Q}$ for their Laplace transforms, and $\overset{\circ}{I}$, $\overset{\circ}{V}$, $\overset{\circ}{Q}$ for their values * when $t=0$.

We have to solve †

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = V, \quad t>0, \quad \ldots\ldots\ldots\ldots(15)$$

where Q, the charge on the condenser, is given by

$$Q = \overset{\circ}{Q} + \int_0^t I(t')\,dt'. \quad \ldots\ldots\ldots\ldots(16)$$

Multiplying (15) by e^{-pt} and integrating with respect to t from 0 to ∞ we obtain, as in § 3,

$$(Lp+R)\bar{I} + \frac{\bar{Q}}{C} = \bar{V} + L\overset{\circ}{I}. \quad \ldots\ldots\ldots\ldots(17)$$

Similarly from (16) we find

$$\begin{aligned}\bar{Q} &= \frac{\overset{\circ}{Q}}{p} + \int_0^\infty e^{-pt}\,dt \int_0^t I(t')\,dt' \\ &= \frac{\overset{\circ}{Q}}{p} - \left[\frac{1}{p}e^{-pt}\int_0^t I(t')\,dt'\right]_0^\infty + \frac{1}{p}\int_0^\infty e^{-pt} I(t)\,dt \\ &= \frac{\overset{\circ}{Q}}{p} + \frac{\bar{I}}{p}. \quad \ldots\ldots\ldots\ldots(18)\end{aligned}$$

* I owe this notation to Professor Bailey of Sydney University. It will be found very convenient in manuscript.

† I is taken positive when flowing towards the high-potential side of the condenser.

Substituting this in (17) we have the fundamental equation *

$$\left(Lp+R+\frac{1}{Cp}\right)\bar{I}=\bar{V}+L\overset{\circ}{I}-\frac{\overset{\circ}{Q}}{Cp}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(19)$$

A complicated electrical network may be divided into elements of the above type (*i.e.* L, R, C in series) and equations corresponding to (19) written down for each of them ; in this way sufficient equations may be obtained to determine all the currents. In the more general case in which there are mutual inductances M_r between the element considered and the elements $r=1, \ldots, n$, in which the curents are I_r, the argument above gives in place of (19)

$$\left(Lp+R+\frac{1}{Cp}\right)\bar{I}+\sum_{r=1}^{n} pM_r\bar{I}_r=\bar{V}+L\overset{\circ}{I}-\frac{\overset{\circ}{Q}}{Cp}+\sum_{r=1}^{n} M_r\overset{\circ}{I}_r. \quad \ldots(20)$$

5. *Constant E.M.F., E, applied at $t=0$ to the circuit of* § 4. *Initial charge and current zero.*

Here $\overset{\circ}{I}=\overset{\circ}{Q}=0$, $\bar{V}=E/p$, thus (19) becomes

$$\left(Lp+R+\frac{1}{Cp}\right)\bar{I}=\frac{E}{p},$$

and so
$$\bar{I}=\frac{E}{L\left(p^2+\frac{R}{L}p+\frac{1}{LC}\right)}=\frac{E}{L\left[(p+\mu)^2+n^2\right]}, \quad \ldots\ldots\ldots\ldots\ldots(21)$$

where
$$\mu=\frac{R}{2L}, \quad n^2=\frac{1}{LC}-\frac{R^2}{4L^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(22)$$

Therefore if $n^2>0$ we find from (4) and the Theorem (*i.e.* (10))

$$I=\frac{E}{nL}e^{-\mu t}\sin nt.$$

If $n=0$, by (3) and the Theorem

$$I=\frac{E}{L}te^{-\mu t}.$$

If $n^2<0$, putting $n^2=-k^2$, we find

$$I=\frac{E}{kL}e^{-\mu t}\sinh kt.$$

* This was given by Pipes [*Phil. Mag.* (7), 24 (1937), 502], but he obtains $I(t)$ from $\bar{I}$ by contour integration; this is an unnecessary complication when only ordinary differential equations are being considered.

6. *Condenser charged to potential E and discharged at $t=0$ through an inductive resistance.*

Here $\overset{\circ}{I}=0$, $\overset{\circ}{Q}=CE$, $V=0$, and (19) becomes

$$\left(Lp+R+\frac{1}{Cp}\right)\bar{I}=-\frac{E}{p}.$$

Thus in the notation of (22)

$$\bar{I}=-\frac{E}{L[(p+\mu)^2+n^2]}$$

and, as in § 5, $I=-\dfrac{E}{nL}e^{-\mu t}\sin nt$, if $n^2>0$, and if $n^2\leqslant 0$ the results are modified as in § 5.

To find the charge on the condenser we have by (18)

$$\bar{Q}=\frac{CE}{p}-\frac{E}{Lp\left(p^2+\dfrac{R}{L}p+\dfrac{1}{LC}\right)}=\frac{CE(p+2\mu)}{(p+\mu)^2+n^2}.$$

Therefore $$Q=\frac{CE}{n}e^{-\mu t}(\mu\sin nt+n\cos nt).$$

7. *Alternating E.M.F. $E\sin\omega t$ applied at $t=0$ to the circuit of* § 4. *Initial charge and current zero.*

Here $$\overset{\circ}{I}=\overset{\circ}{Q}=0,\quad \bar{V}=\frac{\omega E}{p^2+\omega^2}.$$

Hence by (19), with the notation of (22) (it is assumed that $n^2>0$),

$$\bar{I}=\frac{\omega pE}{L(p^2+\omega^2)[(p+\mu)^2+n^2]}. \qquad \text{.....(23)}$$

Expressing this in partial fractions with quadratic denominators we find

$$\bar{I}=\frac{E}{Z^2}\left\{\frac{X(p+\mu)-X'\mu}{(p+\mu)^2+n^2}-\frac{Xp-R\omega}{p^2+\omega^2}\right\},$$

where $$X=L\omega-\frac{1}{C\omega},\quad X'=L\omega+\frac{1}{C\omega},\quad Z^2=X^2+R^2.$$

Therefore

$$I=\frac{E}{nZ^2}e^{-\mu t}(nX\cos nt-\mu X'\sin nt)-\frac{E}{Z^2}(X\cos\omega t-R\sin\omega t)$$

$$=\frac{E}{Z}\sin(\omega t-\gamma)-\frac{E}{nZ\sqrt{(LC)}}e^{-\mu t}\sin(nt-\delta), \qquad \text{.....(24)}$$

where $\tan\gamma=X/R$, $\tan\delta=nX/(\mu X')$.

Instead of expressing $\bar{I}$ in partial fractions with quadratic denominators it could have been expressed by (14) in partial

fractions with linear (complex) denominators. This process is specially convenient if only the steady state current is required; in this case only the terms corresponding to the roots $\pm i\omega$ of the denominator of (23) need be evaluated, since those corresponding to the roots $-\mu\pm$ *in* give rise to transient terms.

8. *Alternating E.M.F. of the same period and damping applied at* $t=0$ *to the circuit of* § 4 *with zero initial charge and current; to find the subsequent charge on the condenser.*

In the notation of (22) we take $V=Ee^{-\mu t}\sin(nt+\alpha)$, where α is a constant phase angle.

Thus $$\bar{V}=E\frac{(p+\mu)\sin\alpha+n\cos\alpha}{(p+\mu)^2+n^2}.$$

And from (18) and (19) with $\overset{\circ}{I}=\overset{\circ}{Q}=0$,

$$\bar{Q}=\frac{E}{L}\frac{(p+\mu)\sin\alpha+n\cos\alpha}{[(p+\mu)^2+n^2]^2}.$$

Therefore by (8) and (9) and the Theorem

$$Q=\frac{E}{L}e^{-\mu t}\left\{\frac{1}{2n}t\sin nt\sin\alpha+\frac{1}{2n^2}(\sin nt-nt\cos nt)\cos\alpha\right\}$$

$$=\frac{E}{2n^2L}e^{-\mu t}\{\sin nt\cos\alpha-nt\cos(nt+\alpha)\}.$$

9. *Steady current* E/R *is flowing in the circuit of Fig.* 1 *with the switch S closed. At* $t=0$ *the switch S is opened; to find the subsequent current.*

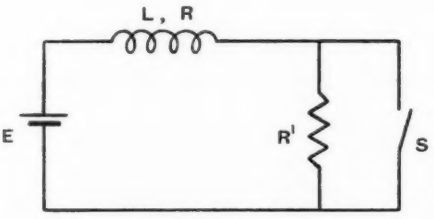


FIG. 1.

Here $\overset{\circ}{I}=E/R$, $V=E$ and (19) gives

$$[Lp+(R+R')]\,\bar{I}=\frac{E}{p}+\frac{EL}{R}.$$

Therefore $$\bar{I}=\frac{E(Lp+R)}{Rp[Lp+R+R']},$$

and so $$I=\frac{E}{R+R'}+\frac{ER'}{R(R+R')}e^{-t(R+R')/L}.$$

10. *Alternating current due to E.M.F.* $V = E\sin(\omega t + \phi)$ *is flowing in the circuit of Fig. 2 with the switch S closed. At* $t = 0$, *when steady state conditions are supposed to have been attained, the switch S is opened. It is required to find the subsequent potential difference between A and B.*

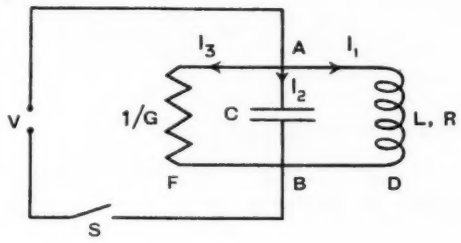


FIG. 2.

Let I_1, I_2, I_3 be the currents in ADB, AB, and AFB, respectively. It follows as in § 7 that the steady state current in ADB due to the E.M.F. $E\sin(\omega t + \phi)$ is $(E/Z)\sin(\omega t + \phi - \gamma)$ where $Z^2 = R^2 + L^2\omega^2$, and $\tan\gamma = L\omega/R$. The value of this when $t = 0$, namely

$$\overset{\circ}{I}_1 = \frac{E}{Z}\sin(\phi - \gamma) \quad \text{.....(25)}$$

is to be used as the initial value of the current in L. Also the charge on the condenser when $t = 0$ is

$$\overset{\circ}{Q}_2 = CE\sin\phi. \quad \text{.....(26)}$$

If v is the potential difference between A and B for $t > 0$, we have from (19) for the circuit elements ADB, AB, AFB, respectively, with initial conditions (25) and (26)

$$\left.\begin{aligned} (Lp + R)\bar{I}_1 &= \bar{v} + \frac{LE}{Z}\sin(\phi - \gamma) \\ \frac{1}{Cp}\bar{I}_2 &= \bar{v} - \frac{E}{p}\sin\phi \\ \frac{1}{G}\bar{I}_3 &= \bar{v} \end{aligned}\right\}. \quad \text{.....(27)}$$

Now $I_1 + I_2 + I_3 = 0$, when $t > 0$, and thus $\bar{I}_1 + \bar{I}_2 + \bar{I}_3 = 0$. Using this result in (27) we find

$$\bar{v}\left\{\frac{1}{Lp + R} + Cp + G\right\} - CE\sin\phi + \frac{LE\sin(\phi - \gamma)}{Z(Lp + R)} = 0.$$

Thus

$$\bar{v} = \frac{CEZ(Lp+R)\sin\phi - LE\sin(\phi-\gamma)}{Z[(Lp+R)(Cp+G)+1]}$$

$$= \frac{LCEZ(p+\alpha)\sin\phi + E\{\frac{1}{2}Z(RC-LG)\sin\phi - L\sin(\phi-\gamma)\}}{LCZ[(p+\alpha)^2+\beta^2]},$$

where $$\alpha = \tfrac{1}{2}\left(\frac{R}{L}+\frac{G}{C}\right), \quad \beta^2 = \frac{RG+1}{LC} - \alpha^2.$$

Hence

$$v = E\sin\phi e^{-\alpha t}\cos\beta t + \frac{E}{\beta LCZ}e^{-\alpha t} \times \{\tfrac{1}{2}Z(RC-LG)\sin\phi - L\sin(\phi-\gamma)\}\sin\beta t.$$

THE UNIVERSITY OF TASMANIA. J. C. J.

CORRESPONDENCE.

WHAT IS STANDARD FORM?

To the Editor of the *Mathematical Gazette*.

DEAR SIR,—Mathematical Association Reports and many text-books say that "a number in standard form is one in which there is one and only one digit to the left of the decimal point". Thus 3·1416 is in standard form.

Nunn in his *Algebra* says: "A number is expressed in standard form when it is written as a decimal with a single digit before the decimal point multiplied by some positive or negative power of 10. Thus the standard form of the numbers 18574 and 0·0005937 are respectively $1{\cdot}8574 \times 10^4$ and $5{\cdot}937 \times 10^{-4}$". I have also found one arithmetic book which uses the phrase in the same sense.

Can any of your readers say which meaning of the expression is correct? And what is its history?

The earliest mention of standard form that I have found in the *Mathematical Gazette* is in Vol. IV, p. 141, in the Teaching Committee's Report on the Teaching of Mathematics in Preparatory Schools. There it is used in the sense I have mentioned first, that is, a number with one digit to the left of the decimal point.

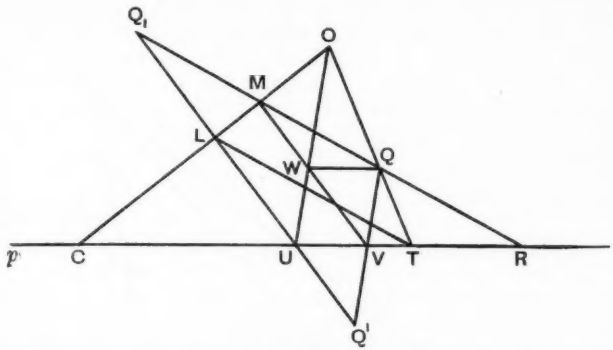
Yours truly,

A. W. SIDDONS.

1309. It is a common custom to refer to the usual complication between one man and two ladies, or one lady and two men, or a lady and a man and a nobleman, or—well, any of those problems—as the triangle. But they are never unqualified triangles. They are always isosceles—never equilateral. So, upon the coming of Nevada Warren, she and Gilbert and Barbara Ross lined up into such a figurative triangle; and of that triangle Barbara formed the hypotenuse.—O. Henry, *Schools and Schools*. [Per Mr. F. J. Wood.]

MATHEMATICAL NOTES.

1435. *Notes on Conics. 2: Radial Measures and the Orthoptic Constant.*



1. Let p be a diameter of a conic, let QVQ' be a double ordinate to this diameter from a point Q of the conic, and let the tangent at Q cut p in T. If Q_1 is any other point of the conic, let Q_1Q' cut p in U, let M be the midpoint of QQ_1, and let the ordinate through U to p cut the line through Q parallel to p in W. Then the line VW, since it bisects QQ' and QU, is parallel to $Q'U$ and bisects QQ_1 also, that is, passes through M. If then the tangent QT cuts UW in O and the line OM cuts UQ_1 in L, we have

$$OM/OL = OW/OU = OQ/OT$$

and LT is parallel to MQ. But the tangents at Q and Q_1 meet on the ordinate to p through U, that is, meet at O, and therefore the line OM passes through the centre C, and if Q_1Q cuts p in R we have $CU/CV = CL/CM = CT/CR$, whence

$$CV \,.\, CT = CU \,.\, CR,$$

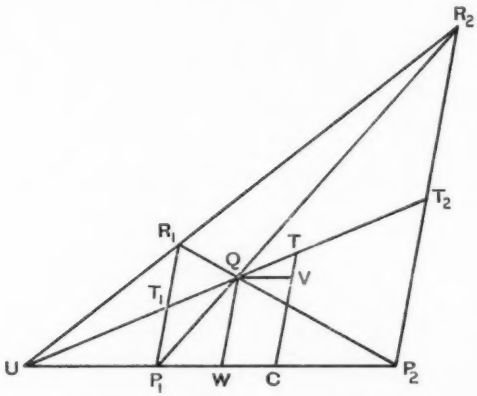
and if the tangent at Q_1 cuts p in T_1, the product $CV_1 \,.\, CT_1$ has this same value.

That is to say, for a given diameter p the product $CV \,.\, CT$ is independent of the position of Q on the conic; it is a constant associated with the diameter. If p cuts the conic in a point P, the constant is immediately identified with CP^2. We may therefore call the constant the radial measure along p, but we have to understand that it is no less significant a constant when negative than when positive.

2. Let P_1, P_2 be diametrically opposite points of a conic, let Q be any other point of the conic, and let the tangent at Q cut the tangents at P_1, P_2 in T_1, T_2 and cut the diameter P_1P_2 in U. Since

T_1C bisects both P_1P_2 and P_1Q, this line is parallel to P_2Q, and if P_2Q cuts P_1T_1 in R_1, then T_1 is the midpoint of P_1R_1; similarly if P_1Q cuts P_2T_2 in R_2, then T_2 is the midpoint of P_2R_2. It follows that R_1R_2 passes through U, that U, Q and T_1, T_2 are harmonic pairs, and that if T is the midpoint of T_1T_2, then

$$UT_1 \cdot UT_2 = UQ \cdot UT.$$



But CT is parallel to P_1T_1 and P_2T_2, and if the line through Q parallel to these lines cuts CU in W, the lengths P_1T_1, P_2T_2, WQ, CT are proportional to the lengths UT_1, UT_2, UQ, UT, and therefore $P_1T_1 \cdot P_2T_2 = WQ \cdot CT$; drawing QV parallel to CU to meet CT in V, we have $CV = WQ$, and we recognise in $CV \cdot CT$ the radial measure of the diameter CT. *The product of the intercepts on two parallel tangents by another tangent is the radial measure of the diameter which is parallel to the parallel tangents.* If CT_2 cuts P_1T_1 in T_1', we may replace P_2T_2 by $T_1'P_1$, and since CT_1, CT_2 may be described simply as two conjugate diameters the result takes the form that *If a pair of conjugate diameters of a conic meet the tangent at a point P in T', T'', the product $T'P \cdot PT''$ is the radial measure of the diameter parallel to the tangent,* or, in other words, of the diameter conjugate to CP.

3. With the notation of the last paragraph, the chords QP_1, QP_2 are ordinates respectively to the diameters CT_1, CT_2, and if these chords meet their diameters in V_1, V_2, the products $CV_1 \cdot CT_1$, $CV_2 \cdot CT_2$ are the radial measures of the two diameters. From the collinear points T_1, T_2, Q we have

$$QT_2 \cdot CT_1^2 + T_1Q \cdot CT_2^2 = T_1T_2 \cdot CQ^2 + T_1T_2 \cdot T_1Q \cdot QT_2,$$

that is,

$$(QT_2/T_1T_2)\, CT_1^2 + (T_1Q/T_1T_2)\, CT_2^2 = CQ^2 + T_1Q \cdot QT_2.$$

But $QT_2/T_1T_2 = CV_1/CT_1$, $T_1Q/T_1T_2 = CV_2/CT_2$; hence

$$CV_1 \cdot CT_1 + CV_2 \cdot CT_2 = CQ^2 + T_1Q \cdot QT_2.$$

That is to say, the sum of the radial measures of the two conjugate diameters CT_1, CT_2 is equal to the sum of the radial measures of CQ and the diameter conjugate to CQ. Hence *the sum of the radial measures of two conjugate diameters is the same for all pairs of conjugate diameters*, and this sum is a constant of the conic. We can associate the constant at once with its most familiar property, for since T is the midpoint of T_1T_2, the product $T_1Q \cdot QT_2$ is equal to $TT_1^2 - TQ^2$, and if the tangents T_1P_1, T_1Q are perpendicular, $CT_1^2 = CT^2 + CQ^2 - TQ^2$. The sum of the radial measures of conjugate diameters may therefore appropriately be called the orthoptic constant of the conic.

4. If S, S' are isogonal conjugates in a triangle ABC, the diameter d of the circle through the feet of the perpendiculars from S, S' to the sides is given by the theorem that if H is the orthocentre, $d^2 - SH^2 - S'H^2$ has the same value for all conjugate pairs, namely the value $R^2 - OH^2$ to which it reduces when S, S' are O, H and the pedal circle is the nine-point circle. In terms of the inscribed conic of which S, S' are the foci, the pedal circle is the auxiliary circle, and if K is the midpoint of SS', $d^2 - 2KS^2$ is twice the orthoptic constant ω; hence $d^2 - SH^2 - S'H^2 = 2(\omega - HK^2)$. On the other hand, if AH cuts BC in D and cuts the circumcircle again in H_1, the reflection of H in BC, then $OH^2 - R^2$, the power of H for the circumcircle, is equal to $HA \cdot HH_1$, that is, to $2HA \cdot HD$; the product $HA \cdot HD$ is the polar constant ρ of the triangle, which is the square of the radius of the polar circle if the triangle is obtuse-angled. Thus $HK^2 - \omega = \rho$:

If a triangle circumscribes a conic, the square of the distance between the orthocentre of the triangle and the centre of the conic is the sum of the polar constant of the one and the orthoptic constant of the other.

If one of the constants involved is positive, the theorem can be expressed in terms of the corresponding circle:

If a conic is inscribed in an obtuse-angled triangle, the orthoptic constant of the conic is the power of the centre for the polar circle of the triangle;

If a triangle circumscribes an ellipse or an acute hyperbola, the polar constant of the triangle is the power of the orthocentre for the orthoptic circle of the conic.

If the constants are both positive, we have a theorem more strictly geometrical in form:

If an obtuse-angled triangle circumscribes an ellipse or an acute hyperbola, the polar circle of the triangle is orthogonal to the orthoptic circle of the conic.

That the last theorem has a limiting form applicable to the parabola is obvious; to say that the polar circle is orthogonal to the

directrix is only to assert that the directrix passes through the orthocentre. That this result is true whether or not the triangle is obtuse-angled actually follows from the preceding theorem, if we suppose the triangle to be fixed and the conic to tend to a parabola, for if the radius of a circle tends to infinity, the power of a fixed point for the circle tends to infinity also unless the circle in its limiting form passes through the point. E. H. N.

1436. *On Note* 1395 (XXIII, July 1939, p. 301).

As solutions of the equation $x^2+z^2-2y^2=0$ we have

$$(m^2-2l^2)^2+(m^2+2l^2-4lm)^2-2(m^2+2l^2-2lm)^2=0,$$

Journal de Mathématiques élémentaires (Bourget et de Longchamps), 1895, p. 12, Boutin ; *Sphinx-Oedipe*, 1906, p. 95, Gérardin.

$$\{(l+m)^2-2l^2\}^2+\{(l+m)^2-2m^2\}^2-2(l^2+m^2)^2=0,$$

Journal de Mathématiques élémentaires (Bourget et de Longchamps), 1897, p. 95, Plakhowo ; *L'Intermédiaire des Mathématiciens*, 1909, p. 19, Welsch.

$$\{(p+3q)^2-2q^2\}^2+\{(p+q)^2-2q^2\}^2-2\{(p+2q)^2+q^2\}^2=0$$
$$\{(p-3q)^2-2q^2\}^2+\{(p+q)^2-2p^2\}^2-2\{(p-2q)^2+q^2\}^2=0,$$

L'Intermédiaire, 1908, p. 102, Tafelmacher.

See also *L'Intermédiaire*, 1908, p. 259 ; 1909, p. 156, p. 254 ; 1910, p. 36 : Fitzpatrick, *Exercices d'Arithmétique*, p. 100, No. 660 (3rd edition) : *Zeitschrift für Math. und Naturw. Unterricht*, 61, 1930, p. 460.

Gérardin (*L'Intermédiaire*, 1909, p. 156) also gives the following references : Ed. Lucas, *Recherches sur l'analyse indéterminée*, 1873, p. 33, and *Bollettino di Bibliografia e Storia delle Scienze matematiche e fiziche* (pubblicato del P. B. Boncompagni), 1876, p. 591 ; 1877, p. 187 ; Abria et Hoüel, *Notice sur la vie et les travaux de V.-A. Le Besgue*, p. 40 ; Legendre, *Théorie des nombres*, ii, p. 126 ; Ed. Lucas, *Nouvelles Annales de Mathématiques*, 1878, p. 446. J. MAHRENHOLZ.

1437. *Feuerbach's Theorem.*

It is a great pity that so beautiful and striking a theorem as Feuerbach's has formerly been available to so small a number of students because of the difficulty of its proofs, (or, on a lower intellectual level, of even demonstrating it by drawing). W. J. D.'s simple proofs (Note 1387, last July), which should do much to remedy this limitation, have stimulated me to try and draw a decent figure in which all the contacts and intersections are shown distinct from each other. Most of us find that such efforts, if made by fair means, fail badly, either because our triangles are too nearly isosceles or because of the cumulative errors due to the evil spirits which inhabit our instruments.

The use of squared paper and calculated coordinates, while not doing away with errors, does at any rate destroy their cumulative

character: a further advantage is that when the figure still fails to come up to expectation we can put the blame on to the irregularity of the squaring instead of our own incompetence. I am therefore giving some coordinates, etc., mostly integral, from which any schoolboy, if he has big enough compasses, can make a distinct and convincing figure to show all the contacts and intersections clearly separated. With a tenth-inch unit, all the essentials of the figure can fit on to a piece of squared paper $17'' \times 21''$, costing $1\frac{1}{2}$d.

$a = 84$	$s = 105$	$r = 18$
$b = 75$	$\Delta = 1890$	$r_a = 90$
$c = 51$	$R = 42\frac{1}{2}$	$r_b = 63$
		$r_c = 35$

	A	B	C
sin	84/85	15/17	3/5
cos	13/85	8/17	4/5
tan	84/13	15/8	3/4
tan $\frac{1}{2}\angle$	6/7	3/5	1/3

		x	y
	A	60	45
	B	84	0
	C	0	0
Orthocentre -	P	60	32
Circumcentre -	O	42	$6\frac{1}{2}$
9-pts.-centre -	U	51	$19\frac{1}{4}$
Incentre -	I	54	18
Ecentres -	I_a	30	−90
	I_b	−21	63
	I_c	105	35

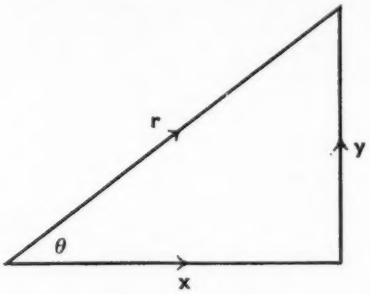
For the benefit of any heroes who after this may still want to draw twelve more circles touching the Nine-points-circle, here are the centres and radii of the inscribed and escribed circles of the triangles PBC, PCA and PAB. For these a bigger piece of paper will be needed.

PBC x	y	radius	PCA x	y	radius	PAB x	y	radius
56	14	14	55	35	5	63	33	3
28	−112	112	−70	110	130	216	84	156
−12	48	48	21	−33	39	73	−7	13
96	24	24	66	42	6	56	44	4

W. HOPE-JONES.

1438. *Note on Note* 1409.

Do technical students learn about vectors? If so, they, and other people, might prove the addition theorem as follows:



If $\mathbf{r}_1 = \mathbf{x}_1 + \mathbf{y}_1$ and $\mathbf{r}_2 = \mathbf{x}_2 + \mathbf{y}_2$,

then $r_1 r_2 \cos(\theta_1 - \theta_2) = x_1 x_2 + y_1 y_2$,

and so $\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$.

A. R.

1439. *A theorem on exponentials and series.*

Let $f(n)$ be of degree r in n and write

$$f(n) = A_0 + \sum_{s=1}^{r} A_s \,.\, n(n-1)(n-2)\ldots(n-s+1)\,; \quad \ldots\ldots\ldots\ldots\text{(i)}$$

then
$$\sum_{n=0}^{\infty} \frac{f(n)}{n!} = A_0 \sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=1}^{\infty} \sum_{s=1}^{r} \frac{n(n-1)\ldots(n-s+1)\,.\,A_s}{n!}$$

$$= A_0\, e + \sum_{s=1}^{r} A_s \sum_{n=s}^{\infty} \frac{1}{(n-s)!}$$

$$= (A_0 + A_1 + A_2 + \ldots + A_r)\, e. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

We can prove that for $s = 1, 2, \ldots, r$

$$s!\, A_s = \{f(s) - {}_sC_1 f(s-1) + {}_sC_2 f(s-2) \ldots + (-)^s f(0)\}. \quad \ldots\text{(iii)}$$

For, if $n = 0, 1, 2$ in (i), we have

$$f(0) = A_0,$$
$$f(1) = A_0 + A_1,$$
$$f(2) = A_0 + 2A_1 + 2A_2,$$

and so

$$A_0 = f(0),$$
$$A_1 = f(1) - f(0),$$
$$2!\, A_2 = f(2) - 2f(1) + f(0).$$

Assume then that for $p = 1, 2, \ldots, s-1$,

$$p!\, A_p = f(p) - {}_pC_1\, f(p-1) + {}_pC_2 f(p-2) \ldots + (-)^p f(0). \quad \ldots(\text{iv})$$

Then as the coefficients in A_p are those of $(f-1)^p/p!$, let us write symbolically

$$f(p) \equiv f^p,\ f(0) \equiv f^0 = 1,$$
$$p!\, A_p = (f-1)^p. \quad (p = 1, 2, \ldots, s-1).$$

Then $f^s \equiv f(s)$

$$= A_0 + sA_1 + s(s-1)A_2 + s(s-1)(s-2)A_3 + \ldots + s(s-1)\ldots 3 \,.\, 2 \,.\, A_{s-1} + s!\, A_s$$
$$= 1 + {}_sC_1(f-1) + {}_sC_2(f-1)^2 + \ldots + {}_sC_{s-1}(f-1)^{s-1} + s!\, A_s$$
$$= \{1 + (f-1)\}^s - (f-1)^s + s!\, A_s$$
$$= f^s - (f-1)^s + s!\, A_s.$$

Hence $s!\, A_s = (f-1)^s$

$$= f(s) - {}_sC_1 f(s-1) + {}_sC_2 f(s-2) \ldots + (-)^s f(0)$$

and (iii) is proved by induction.

By substituting (iii) and (ii) we obtain

$$\sum_{n=0}^{\infty} \frac{f(n)}{n!} = e \sum_{s=0}^{r} \frac{f(s)}{s!} \left\{ \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} \ldots + \frac{(-)^{r-s}}{(r-s)!} \right\}.$$

By an almost identical proof we can show that if

$$f(n) = B_0 + \sum_{s=1}^{n} n(n+1)\ldots(n+s-1)\, B_s,$$

then

$$s!\, B_s = (-)^s \{ f(-s) - {}_sC_1 f(-s+1) + {}_sC_2 f(-s+2) - \ldots + (-)^s f(0) \}.$$

This is useful in connection with the formula

$$\sum_{n=1}^{m} n(n+1)\ldots(n+s-1) = m(m+1)\ldots(m+s)/(s+1),$$

for then

$$\sum_{n=1}^{m} f(n) = mB_0 + \sum_{n=1}^{m} \sum_{s=1}^{r} n(n+1)\ldots(n+s-1)\, B_s$$
$$= mB_0 + \sum_{s=1}^{r} m(m+1)\ldots(m+s)\, B_s/(s+1).$$

A. J. Carr.

1440. *Uniform convergence.*

Quite good students are often puzzled by uniform convergence; their trouble with the definition usually given is to understand what it is all about. On the other hand, they can take in easily, with the help of simple examples, what is meant by one series (or infinite integral) converging more rapidly than another. On this basis one

can construct a definition which (in my experience at any rate) students find it much easier to assimilate. Thus :

(i) If S_n is the sum of the first n terms of a series S, and

$$S = S_n + r_n, \quad S' = S'_n + r'_n,$$

the series S is said to converge more rapidly than S' if

$$|r_n| < |r'_n|$$

for all values of n.

If we were only concerned with establishing a criterium of relative speed of convergence, it would be natural only to require that this inequality should hold for values of n greater than some number n_0; but for our purpose there is an advantage in imposing the more restrictive condition.

(ii) If the terms of a series are functions of one or more variables, the series is said to be uniformly convergent for a given range of values of these variables, if for every set of values of the variables within the given range, the series converges at least as rapidly as some *fixed* convergent series.

Thus, if the terms of the series are functions of one variable x, you have non-uniform convergence for a given value of x, if the rate of convergence tends to be infinitely slow as x approaches the given value.

The definition is easily seen to be equivalent to that usually given. The reason for imposing the more stringent condition in (i) is that the number n_0 would introduce an element of complication in (ii). Would n_0 have to be chosen independently of the variables on which the values of the terms of the series depend? It is much better to be rid of it altogether, and nothing is lost from the point of view of uniform convergence. M. F. EGAN.

1441. *The fundamental property of a Farey series.*

The familiar theorem, that if a, b, c, d are positive integers such that $ad \sim bc = 1$, then no fraction with denominator smaller than $b+d$ can lie between a/b and c/d, is not usually accompanied by its converse, but this theorem also is true :

If a, b, c, d *are positive integers such that* $\mathrm{ad} \sim \mathrm{bc} > 1$, *and if the fractions* a/b *and* c/d *are in their lowest terms, then there is at least one fraction between these two with a denominator smaller than* max (b, d).

Suppose that $b \geqslant d$, and let x, y be the unique pair of numbers such that $ay - bx = 1$, $x < a$, $y < b$; then also $(a-x)b - (b-y)a = 1$. There is no fraction between a/b and x/y on the one side, or between a/b and $(a-x)/(b-y)$ on the other side, whose denominator is not greater than b, and therefore c/d is not within either of these intervals. That is to say, if c/d is less than a/b, then x/y is between them, and if c/d is greater than a/b, then $(a-x)/(b-y)$ is between them.

It follows at once, and surely the argument is simpler than either of the three given in Hardy and Wright's *Theory of Numbers*, that a/b, c/d cannot be consecutive terms of a Farey series unless $ad \sim bc = 1$.

E. H. N.

1442. *Pythagorean numbers.*

Some time ago I came across a formula leading to the construction of the table of Pythagorean numbers given in Note 1391 in the July *Gazette*. I was asked by a class, familiar with (3, 4, 5), (5, 12, 13), (7, 24, 25), if a set could always be constructed where one number is odd and the other two differ by unity. In reply I suggested that $(2n+1,\ x,\ x+1)$ be considered and we soon found that $(2n^2+2n+1,\ 2n^2+2n,\ 2n+1)$ is a set of Pythagorean numbers.

In class we had only considered n to be an integer, but on substituting p/q for n and multiplying by q^2, we have the set

$$(2p^2+2pq+q^2,\ 2p^2+2pq,\ 2pq+q^2). \quad \ldots\ldots\ldots\ldots\ldots\ldots (A)$$

Taking p and q to be any two integers, the value of $2p^2+2pq+q^2$ gives the hypotenuse, and the other sides are obtained by subtracting from this (i) $2p^2$, (ii) q^2.

By putting $p+q=m$ and $p=n$, the formula (A) is transformed to the well-known

$$(m^2+n^2,\ m^2-n^2,\ 2mn), \quad \ldots\ldots\ldots\ldots\ldots\ldots (B)$$

but (A) leads to the table mentioned above and I know of no simpler one formed by using (B).

If we use (A) we need not consider even values of q. For, if we write $2q$ for q, we have $(2p^2+4pq+4q^2,\ 2p^2+4pq,\ 4pq+4q^2)$ and hence $(p^2+2pq+2q^2,\ p^2+2pq,\ 2pq+2q^2)$, which is (A) with p and q interchanged.

The table then follows :

Arrangement (a).

q	q^2	p = $2p^2$ =	1	2	3
			2	8	18 (multiply the above by 2, 4, 6, ...)
1	1	$2p^2+2p+1=$	5	13	25 (add to above $2p+1=3, 5, 7, \ldots$)
3	9	$2p^2+6p+9=$	17	29	45 (add to above $4p+8=12, 16, 20, \ldots$)
5	25	$2p^2+10p+25=$	37	53	73 (add to above $4p+16=20, 24, 28, \ldots$)
...	...	.....	...	...	... .

Arrangement (b).

1	5	13	25	41	61	(as above)
	9	17	29	45	65	(by adding 4 to each above)
		25	37	53	73	(by adding 8 to each above)
			49	65	85	(by adding 12 to each above)
				81	101	(by adding 16 to each above)
					121	(by adding 20 to each above).

E. McLean.

REVIEWS

The philosophy of physical science. By SIR ARTHUR EDDINGTON. Pp. ix, 230. 8s. 6d. 1939. (Cambridge)

The weight of responsibility on a reviewer of this book is lightened by the realisation that nowadays nobody waits for his verdict before reading something by Sir Arthur Eddington. And, moreover, Eddington's readers know better than to expect a reviewer to produce anything that might fairly be called a verdict. If they trouble about him at all, it is probably merely in order to compare notes with him about first impressions of Eddington's ideas.

However, should anyone chance to see this notice before meeting with the book itself, he may be assured, if assurance be needed, that in no respect does it fall short of Eddington's previous writings, either in the fascination of its style, or in the fundamental importance of the problems dealt with, or in the persuasiveness of its arguments. He may also be interested in the fact that, though this book is a natural outcome of the previous writings, particularly *The Nature of the Physical World*, it is developed with a change of emphasis. Whereas the earlier books were concerned in the first place with what we have come to know of the physical universe, this one is primarily concerned with an enquiry into the nature of this knowledge itself. This enquiry here leads to the systematic presentation of Eddington's philosophy of physical science, which, in the closing chapters, he relates to a general philosophical outlook upon the nature of man.

Having said this, one may offer for what they are worth a few comments upon some of the ideas presented in the book. To begin with, physical knowledge is taken to mean only knowledge capable of observational test, and so we may express physical knowledge only in terms of quantities whose definitions imply methods of measuring them. It seems easy enough to accept this standpoint, but, as Eddington shows, its non-acceptance in the past has been responsible for a number of difficulties. For instance, absolute simultaneity at a distance, and an unambiguous " distance " between remote bodies, were notions employed in the formulation of classical physical theories, but no methods of observing them were specified. When they were successively discarded we got in turn the theories of special and general relativity. Again, relative coordinates of a pair of particles, of which the definition has a meaning in terms of possible observations only when the particles are distinguishable, have been used in quantum theory in dealing with pairs of electrons, which are postulated to be observationally indistinguishable. To this Eddington traces certain failures of the theory.

Thus an elementary enquiry into the nature of physical knowledge assists the progress of physics itself, in the rather negative way of getting rid of meaningless features. We can further, without much difficulty, follow Eddington as far as to foresee some general features of physical theory which may be expected to result from a proper view of physical knowledge. As already noticed, the definition of a physical quantity must specify how it is to be measured. In ensuring that the conditions under which the quantity is measured are those specified, other quantities will have to be measured, and so on for each set of quantities. In order to avoid an infinite regression of measurements, we have to be prepared for a lack of exactness in the measurements themselves.* This circumstance can be allowed for, in Eddington's

* The unavoidable character of this interference of measurements appears from another standpoint to be due to the discreteness of the quantum of action (Bohr, *Atomic Theory and the Description of Nature*). So, though Eddington's arguments show that we *may* have to give the probability concept a fundamental place in physical theory, it is difficult to admit at this stage that we are *compelled* to do

view, only by incorporating the conception of probability in the fundamental definitions, so we have an example of an epistemological investigation giving us *a priori* knowledge of an essential feature of physical theory.

Even when we have granted all this, we are still, however, only a short way along the road to the central implication of Eddington's philosophy, which is that " when the epistemological scrutiny of definitions is systematically applied, and its consequences are followed up mathematically, we are able to determine all the " fundamental " laws of nature (including the purely numerical constants of nature) without any physical hypothesis. This means that the fundamental laws and constants of physics are wholly subjective ". Hence, incidentally, comes Eddington's first suggestion of *selective subjectivism* as a name for this philosophy—" selective " referring in a general way to the operation of observing acting upon something objective.

Before venturing to make any comment upon this revolutionary claim, we may notice that the mathematical development is given an essential rôle, and so we may look for a moment at Eddington's subsequent discussion of the reason for the introduction of mathematics into physical theory. Eddington says (quoting statements not in the order in which they occur in the book) " The [logical] starting point of physical science is knowledge of *the group-structure of a set of sensations in a consciousness.*" " Physical science consists of purely structural knowledge, so that we know only the structure of the universe which it describes." " Since the structure, abstracted from whatever possesses the structure, can be exactly specified by mathematical formulae, our knowledge of structure is communicable." This implies that, if we use the right mathematical system, its abstract structure exactly matches the structure of the set of sensations which we call physical experience, and that this structure comprises all the features of this experience which one intelligence can share with another. Hence Eddington's alternative suggestion of *structuralism* as a name for his philosophy.

Now it seems to the reviewer that it is in the choice of the " right " mathematical system that the claims of Eddington's subjectivism may be open to criticism. From one's reading of the present book, it appears that he would assert that " when its consequences are followed up mathematically " we should get the mathematical system set out in his *Relativity theory of protons and electrons* (though the latter book did not profess to be developed strictly from this point of view ; rather was the point of view presented as an outcome of that work). He also appears to claim that a correct epistemology leads inevitably to this particular mathematical system.

It is this inevitability which it is so difficult to accept. One can see that it is strongly *suggested* by the epistemological considerations, but not that it can be rigorously deduced from them. We would be ready to experiment with the employment of these considerations, instead of with physical hypotheses or mechanical or geometrical models, as means to suggest a mathematical scheme for physical theory, but having got it in this way, most of us would probably be disinclined to adopt an attitude towards it different from that we adopt towards a mathematical scheme got in one of the other ways. It may well be that we should find Eddington's scheme to yield a " better " theory, but we should judge this by the criteria applied also to other theories by means of comparisons of theoretical predictions with observation.

This attitude towards Eddington's work would remove what appears to be almost an internal inconsistency in it. He says " Epistemological laws (if correctly deduced) are compulsory, universal, and exact ". (The exactness

so. All this is somewhat aside from the general circumstance that every prediction of the future is implicitly a statement of a probability.

may apply in some cases, as he points out, only to statements about probabilities.) But we may fairly ask how laws having such characteristics represent knowledge capable of observational test. The analysis of what we mean by the observational test of a physical law presents a very difficult problem. It may be sufficient, however, to remark that since in practice we merely test its application to a finite number of special cases, within the limits of experimental error, we can never claim to have verified its universality and exactness, and in fact it appears impossible to attach a meaning, in terms of possible observations, to these terms.

The attitude advocated here gives the "laws" a less imposing status, and indeed practically denies that they themselves represent physical knowledge. The physical knowledge at any epoch consists of what we know about physical observations that have actually been made, and part of this knowledge may be that certain special deductions from theoretical laws have been verified to a certain degree of approximation. This brings us back to the beginning by indicating that these comments are based on a view of physical knowledge rather different from Eddington's.

The very tentative spirit of these comments must be stressed. In any case they touch upon just one or two leading ideas amongst the wealth of those contained in the book. All these ideas will be long pondered over by its readers, who assuredly will be legion. W. H. McCrea.

Mr. Tompkins in Wonderland. By G. Gamow. Pp. x, 91. 7s. 6d. 1939. (Cambridge)

We have all had dreams in which the ordinary facts of everyday experience have been curiously distorted. So had Mr. Tompkins. But Mr. Tompkins was singularly fortunate in that the distortions in his dreams were controlled by very definite rules, resulting, in fact, simply from changes in fundamental physical magnitudes. In some of his dreams he visited places where the velocity of light was about 10 miles per hour, other world-constants being the same as in ordinary life. In others he got to places where the quantum-constant h was 1 erg sec. (not 1 erg/sec. as printed) instead of the usual value $6{\cdot}62 \times 10^{-27}$ erg sec. In his first dream he found himself in a universe where the velocity of light was a million times smaller, and the gravitational constant a million million times larger than in the actual universe. It was a pulsating universe with a period of about 2 hours and a maximum radius of about 100 miles, and it was filled with dust having a density of about 1 lb. per cubic mile when the radius reached its maximum value. His last dream was the most alarming, for in it he was having a holiday in a place where physical constants had the unhappy knack of undergoing enormous changes with disconcerting rapidity.

Like most dreams, those of Mr. Tompkins were stimulated by an activity of his waking life. Actually he was attending a course of lectures given by the professor on recent fundamental advances in physics. These lectures are reproduced for us at the end of the book. They reveal great skill on the part of the professor in the art of popular exposition. But, in spite of that, Mr. Tompkins, who spent his days as a bank clerk and had had only negligible scientific training, found them difficult to follow. However, the experiences he had in his dreams, aided by occasional explanations by the professor, who also made his appearance in them at opportune moments, led him to grasp the main ideas with such a degree of confidence that he finally had sufficient assurance to make a proposal of marriage—he was not dreaming at the time he did so—to the professor's daughter.

Thanks to the record of all this set forth by Professor Gamow, aided by the extraordinarily skilful illustrations by John Hookham, readers starting with no more advantages than those possessed by Mr. Tompkins will surely

be as successful as he in arriving at an understanding of fundamental ideas in relativity and quantum theory. Those who make the attempt are assured of a few hours of genuine enjoyment, even if they have not Mr. Tompkins' good fortune to marry the professor's daughter.

We should mention, however, that the professor seems to have had one or two lapses, due doubtless to his effort to make things as simple as possible for Mr. Tompkins. He tells him (p. 41) about the apparent slowing down of atomic vibrations in the solar atmosphere, as observed from the earth. I think he should have explained that an observer in the solar atmosphere would naturally use a system of time-reckoning such that for him there would be no apparent slowing down of these vibrations. Also in describing the red-shift of light from distant nebulae (p. 9), the professor says that the Andromeda nebula "shows 0·05% of reddening". Actually this is what we think it would show to an observer at rest with respect to the centre of gravity of the Milky Way. But what it does show to an observer on the earth is a small *blue*-shift. In this connection, too, the professor has been unjustifiably dogmatic (p. 10) about the pulsation of the actual universe; most workers would not agree with his positive assertion that the expansion of the universe must stop after a certain time.

The reviewer has refrained from recounting any of Mr. Tompkins' dream-adventures lest he should spoil the pleasure of those who will read the book itself. One quotation can, however, scarcely be resisted.

'"But tell me" [said Mr. Tompkins], "if there were no matter, would we have the kind of geometry I was taught at school, and would parallel lines never meet?"

"They would not", answered the professor, "but neither would there be any material creature to check it."

"Well, perhaps Euclid never existed, and therefore could construct the geometry of absolutely empty space?"

But the professor apparently did not like to enter into this metaphysical discussion.'

This seems to indicate a subtlety of mind on the part of Mr. Tompkins which was perhaps after all the reason for his rapid progress in understanding the professor's ideas.

W. H. McCrea.

Probability and Frequency. By H. C. Plummer. Pp. xi, 277. 15s. 1939. (Macmillan)

As recently as 20th June, 1939, Dr. John Wishart, in an address to the Royal Statistical Society (reported in full in their *Journal*, vol. 102, part iv, 1939), regretted that "there appears to be no standard treatise on the theory of statistics treated from the mathematical point of view". He declared that "if the teaching of the subject, at least in relation to its mathematical foundations, is to be seriously discussed, I submit that the question of the text-book must be put right in the forefront. . . ." ". . . It should begin with a study of probability theory, a section which may well prove to be the most difficult of all to write. . . . Thereafter, there is a fairly clear road through distributional theory as an application of probability; the study of distribution functions and their parameters; the theory of estimation (an important section); theories of fitting; tests of goodness of fit; sampling distributions of statistics, leading to tests of significance . . . profusely illustrated by examples; inferences respecting population parameters to be deduced from samples, involving a close examination of fiducial probability; . . . through problems in a single variable to those for many variables, and from methods of dealing with homogeneous data to those concerned with data of a multiply-connected character, the important method known as the analysis of variance occupying a central and commanding place at this stage."

The specifications above of the ideal mathematical treatise on statistics are not easy to comply with, and indeed Dr. Wishart himself considers that it is probably beyond the capacity of any one man to deal with them. In these circumstances, it is the duty of a reviewer to draw particular attention to any book which gives even a part of what is required. There is the last chapter of Uspensky's *Introduction to Mathematical Probability* (reviewed in the *Gazette*, XXII, pp. 202–204, May 1938), but only a very small fraction of this work bears on statistics. There is also a " planigraphed " reproduction of a course of lectures by Dr. Wilks, but it is difficult to obtain it in England. Thus Professor Plummer's book, which he describes as a modest attempt to supply an approach to various topics of probability and statistics from a simple mathematical point of view, will be welcomed. In comparing the contents of his book with the ideal specifications above, it is only fair to remember that the author, in his preface, mentions the omission of several aspects, and claims only to provide an introduction to the general ideas of the subject.

The first chapter deals with probability, introduced at the outset from two points of view, the " equally likely " definition of Laplace and the frequency definition of Chrystal. Professor Plummer says that " it should be considered an important aim of the subject to reconcile these two points of view as far as possible ", but the rest of the chapter, which " is such as might be found in any serious textbook of Algebra ", is concerned exclusively with the application of Laplace's definition. There is no mention of any of the modern developments of the frequency definition, due to von Mises and his school.

The second chapter deals with continuous probability : " it owes more than a little to the suggestive chapter contributed by W. M. Crofton to Williamson's *Integral Calculus* ". For the convenience of students it contains a proof of the elementary properties of Beta and Gamma functions, sufficient for the use of these functions in probability problems.

The third chapter deals with the theory of errors. There is an interesting discussion of the normal law from different points of view. This is followed by an account of the method of least squares.

The titles of the fourth and fifth chapters are somewhat misleading. The fourth chapter is headed " statistical distributions ", but really deals chiefly with curve fitting. After a short treatment of Poisson's sequence, and of practical Fourier analysis, there is an elaborate account of the method of moments, culminating in Pearson's set of generalised probability curves and the methods of fitting empirical data to them.

The fifth and last chapter, headed " correlation ", contains a short discussion of that subject and of the cognate subject regression, but it deals chiefly with statistical distributions. In the reviewer's opinion, it is the most valuable part of the book. It contains the proofs of various theorems due to " Student " (W. S. Gosset), Karl Pearson, and R. A. Fisher. It starts with the extension of the normal law to two, three, and n dimensions, and deduces the distribution of " chi-squared ", with applications to tests of goodness of fit. Then follows an introduction to the theory of small samples drawn from a normal population. The distribution of the mean $\bar{x}$ had been proved to be normal, in a previous chapter. By finding the distribution of $\bar{x}^2+s^2$ there are deduced those of the standard deviation s and of the ratio $\bar{x}/s$. The next distribution given is that of the correlation coefficient r, with an investigation of the cases in which this coefficient may be used in tests of significance. The modern methods of testing for significance are not given. The last section may mislead a careless reader ; it gives the distribution of a variable denoted by t, without warning that it is quite different from what is usually known as the t-distribution. The references include R. A. Fisher's *Statistical Methods*

for Research Workers, but it would have increased the value of Professor Plummer's last chapter if it had been extended (possibly at the expense of the older material in the second chapter) to include the proofs of the methods used by Fisher and others in current practical statistical analysis. In a second edition there should be a much fuller treatment of tests of significance, and some account of the analysis of variance. The lack of numerical examples may perhaps be excused, as these may be found elsewhere.

The author may complain that this review assesses his book by a standard with which it was never meant to comply. But whether intentionally or not, the book does provide an instalment of what is urgently needed, and the student who has learnt statistics from the usual textbooks, in which many important parts are given without proof, will find it well worth while to get Professor Plummer's book in order to obtain some of these proofs, and a grasp of some of the foundations of the subject. H. T. H. PIAGGIO.

Tables of Partitions. By HANSRAJ GUPTA. Pp. vi, 81. Rs. 5. 1939. (Indian Mathematical Society)

A partition of a number n is a representation of n as the sum of any number of positive integral parts. Thus

$$5=4+1=3+2=3+1+1=2+2+1=2+1+1+1=1+1+1+1+1$$

has 7 partitions, since we have, of course, to count the representation by one part only. The order of the parts is irrelevant. The number of partitions is denoted by $p(n)$, so that $p(5)=7$. Euler pointed out that

$$\prod_{s=1}^{\infty}(1-x^s)^{-1}=1+\sum_{n=1}^{\infty}p(n)x^n,$$

and proved that

$$\prod_{s=1}^{\infty}(1-x^s)=1+\sum_{i=1}^{\infty}(-1)^i\{x^{\frac{1}{2}i(3i-1)}+x^{\frac{1}{2}i(3i+1)}\},$$

both of course when $|x|<1$. From these a recurrence formula for $p(n)$ can be found and Macmahon used this to calculate the values of $p(n)$ up to $n=200$.

Professor Gupta has extended this table to $n=600$, using a method of his own which shortens the labour considerably. He first calculates tables of (n, m), the number of partitions of n in which m is the smallest part. This satisfies the recurrence formula

$$(n, m)=(n-m, m)+(n+1, m+1),$$

which leads to a very simple method of forming the tables. Clearly

$$p(n)=(n+1, 1).$$

The work is shortened by the use of simple formulae for (n, m) valid for $n<5m$; thus the tables need only be set out in detail for $m\leqslant[\frac{1}{4}n]$. The tables for (n, m) printed here extend to $m=50$ and $n=300$.

Ramanujan conjectured that $p(n)$ is divisible by q^k, where $q=5$, 7 or 11, whenever $24n-1$ is divisible by q^k. Ramanujan proved this conjecture for $q=5$, 5^2, 7, 7^2, 11 and 11^2. It has since been proved true for $q^k=5^3$, but this table shows that it is false for $q^k=7^3$. In fact, if $n=243$, then

$$24n-1\equiv 0 \pmod{7^3}$$

while $$p(243)=133978259344888 \not\equiv 0 \pmod{7^3}.$$

The introduction to the tables contains an account of Lehmer's recent work on the famous Hardy-Ramanujan series for $p(n)$. E. M. WRIGHT.

Orthogonal Polynomials. By G. Szegö. Pp. ix, 401. $6. 1939. American Mathematical Society Colloquium Publications, 23. (American Mathematical Society, New York)

If $\alpha(x)$ is a non-decreasing function in the interval $[a, b]$, the functions $f(x)$ and $g(x)$ are said to be orthogonal with respect to the distribution $d\alpha(x)$ if their scalar product

$$(f, g)=\int_a^b f(x)g(x)d\alpha(x)$$

vanishes. Neglecting the exceptional case when $\alpha(x)$ has only a finite number of points of increase, and assuming that the " moments "

$$c_n=\int_a^b x^n d\alpha(x) \qquad (n=0, 1, 2, \ldots)$$

exist, we can construct successively functions $p_0(x)$, $p_1(x)$, $p_2(x)$, ... such that

(*a*) $p_n(x)$ is a polynomial of precise degree n in which the coefficient of x^n is positive ;

(*b*) every pair of the polynomials has a zero scalar product with respect to $\alpha(x)$.

These conditions determine the sequence of polynomials up to an arbitrary constant multiplier, which can be fixed by normalization, that is, by requiring that

$$(p_n, p_n)=1. \qquad (n=0, 1, 2, \ldots)$$

A sequence of polynomials determined in this way is called an orthonormal set. It is with such sets of polynomials that Professor Szegö's book is concerned.

Many of the classical polynomials of analysis are of this type and their properties have been investigated in great detail independently of the general theory of orthogonal polynomials and, indeed, even before there was any such general theory. For example, the Legendre polynomials form an orthonormal set over the interval $[-1, 1]$ with respect to the distribution $\alpha(x)=x$; the Laguerre polynomials $L_n^{(\alpha)}(x)$ are orthogonal over the interval $[0, \infty]$ with

$$\alpha(x)=\int_0^x e^{-t}t^{\alpha}dt ;$$

and so on.

The general theory of orthogonal polynomials had its origin in the theory of continued fractions of Stieltjes' type and the associated problem of moments. Although much progress has recently been made in the latter subject, the historical line of approach has been gradually abandoned and the orthogonal property itself has been taken as fundamental ; and this is the point of view which Professor Szegö adopts. He gives a new and detailed development of the main ideas of the theory of orthogonal polynomials and leads up to a very interesting account of recent investigations concerning asymptotic expansions and the distribution of zeros. He applies the general theory to special classes of orthogonal polynomials, in particular those satisfying linear differential equations of the second order, for which more exhaustive results are obtained.

It is impossible in a short review to give a full account of the scope of the work, which may rather inadequately be indicated by the titles of the various chapters, viz. I, Preliminaries ; II, Definition of Orthogonal Polynomials ; III, General Properties ; IV, Jacobi Polynomials ; V, Laguerre and Hermite Polynomials ; VI, Zeros ; VII, Inequalities ; VIII, Asymptotic Properties of the Classical Polynomials ; IX, Expansion Problems associated with the Classical Polynomials ; X, Representation of Positive Functions ; XI, Poly-

nomials orthogonal on the Unit Circle; XII, Asymptotic Properties of General Orthogonal Polynomials; XIII, Expansion Problems associated with General Orthogonal Polynomials; XIV, Interpolation; XV, Mechanical Quadrature; XVI, Polynomials orthogonal on an Arbitrary Curve.

The whole work is characterised by the clarity of exposition and the sustaining of interest which one expects of one of the authors of "Pólya-Szegö". That the book is beautifully printed and singularly free from misprints goes without saying; for it is Volume XXIII of the American Mathematical Society's Colloquium Publications, and fully lives up to the reputation which that series has worthily obtained. E. T. C.

Structure of Algebras. By A. A. ALBERT. Pp. xi, 210. $4. 1939. American Mathematical Society Colloquium Publications, 24. (American Mathematical Society)

Professor Albert is well known for his research in non-commutative algebra and, in particular, in the structure of such algebras, that is, in the relations existing between an algebra and its sub-algebras, a subject to which an increasing amount of attention has been given since the appearance in 1926 of the German edition of L. E. Dickson's *Algebras and their Arithmetics*. His book contains the first complete and connected account of the subject and also a statement of older theories in modern and simplified form. Whereas in these older theories the algebras are taken over the rational, real or complex numbers, in the modern theory the basic field may be algebraic, p-adic or even infinite and quite general. It follows that as a preliminary to the book the reader must be familiar with the ideas underlying the latter as set out clearly in the author's work, *Modern Higher Algebra*,* to which he frequently refers.

The book deals primarily with one of the outstanding achievements of recent years, *i.e.* the determination of all rational division algebras, that is, of algebras taken over an algebraic number field and having no divisors of zero, and the work of R. Brauer, H. Hasse, E. Noether and the author on the subject. The problem is partly arithmetic in character and the solution depends on properties of the p-adic number fields and on the valuation theory, a knowledge of which is now indispensable to algebraists. The excellent account given of these matters in his earlier work is here supplemented by additional results on inseparable fields, splitting fields, p-adic algebras and valuation theory, and a clear, simple and connected account of these rather inaccessible subjects and of the solution of the fundamental problem is now available to the student.

Amongst other notable features of the book is the very good explanation which is given of the theory of crossed products, the importance of which lies in the fact that not only are they the only normal simple algebras which have been actually constructed, but also that every normal simple algebra is equivalent to a crossed product. Indeed it is not yet known whether any normal division algebra exists which is not a crossed product.

Professor Albert treats cyclic algebras as instances of crossed products from which the theory of exponents is deduced as well as the consequent factorisation into direct factors of prime-power degree, thus reversing the usual procedure.

Another subject which the author deals with in an original way is the theory of the representations of algebras by matrix algebras over a field. He applies the results to the theory of Riemannian matrices and their generalisations, the study of which leads on to involutorial algebras to which a whole chapter is devoted.

* Reviewed in *Math. Gazette*, July 1938, XXII, No. 250, pp. 306-7.

Finally, mention must be made of that part of the book which is really an essay in itself on non-commutative algebra and which includes a very comprehensive and most useful bibliography of 468 works which have appeared in recent years, many of which deal with non-associative algebra now becoming important in physics.

The book is excellently produced and will be indispensable to all workers in this field and to others who want an authoritative account of the theory of linear associative algebras. A. R. R.

Determinants and matrices. By A. C. Aitken. Pp. vii, 135. 4s. 6d. 1939. University Mathematical Texts, 1. (Oliver and Boyd)

Even though you consider yourself to be something of an expert in determinants and matrices, you will not read this book while you lounge negligently in an arm-chair. You will, from time to time, be compelled to concentrate while Dr. Aitken presents a topic from an angle that is new to you.

Only an expert could have compressed so much of determinant and matrix theory into so few pages and these pages so very small. There are 132 pages of text and the type on each page occupies a space of less than 4 by 6 inches. But the printing is excellent, the type itself is not unduly small, and the compression is that of clarity and not of confusion of mind.

The reader must, it is true, work and think with the writer but, if he is willing to do this, the matter will be as clear to him as if he had read the fullest and lengthiest explanation that a tiresome teacher could devise; possibly, the matter will be much clearer.

As one would expect, Dr. Aitken has tackled with firmness several details that are only too frequently glossed over. In undergraduate days I swallowed whole certain proofs of the theorem " the value of a determinant is unaltered if rows and columns are interchanged " or, in the brevity of matrix notation, " $|A'| = |A|$ "; and I wrestled valiantly with the signs of a Laplace expansion. It is only in recent years that I have discovered how the crystal-clear expositions of the great algebraists of the nineteenth century have been blurred and marred by their followers. Dr. Aitken thinks out all these matters afresh, and for himself; and so we find " $|A'| = |A|$ " treated as a theorem worthy of some respect and we find the sign of the Laplace expansion worked out with a very welcome firmness of touch. I could wish, though, that Dr. Aitken had gone back to Salmon on this latter point. Salmon's treatment is laconic because the matter is, as he puts it, so obvious: it is only his copyists who have made a mystery of the matter.

The range of the book is admirably described in the publishers' note. The theories of determinants and matrices are developed alternately in such a way that each supports and simplifies the other: the general properties of both determinants and matrices are covered. The book includes a full treatment of rank and of simultaneous equations, while a chapter is devoted to special types of determinants. It contains an excellent account of pivotal condensation, extensional identities, and the use of sub-matrices.

It is a book for every teacher and for every undergraduate who has some first class quality. The weaker brethren will make heavy weather of it unless their hesitant minds are guided by a competent teacher. I do not, for my own part, believe that the admixture of determinant and matrix is a good method for beginners: but my belief is not widely held. Dr. Aitken's book presents that admixture in excellent form and it should find a wide acceptance.

Finally, a gentle jibe: in conformity with his beliefs, Dr. Aitken has placed the matrix first, begins boldly with the matrix in Chapter One, stresses (and rightly stresses) the matrix and the sub-matrix throughout the whole of

the book ; but, in the end, the old Adam of sin has him and the determinant comes into its own. It is when he is writing of determinants that joy flows with the ink from the end of his pen, and when he is composing the title of the book he forgetfully places the determinant first. This friendly jibe is not to belittle the importance of the matrix, but merely to maintain my own private thesis that the matrix has not destroyed the determinant.

W. L. FERRAR.

Integration of ordinary differential equations. By E. L. INCE. Pp. viii, 148. 4s. 6d. 1939. University Mathematical texts, 4. (Oliver and Boyd)

Those who know Dr. Ince's admirable *Ordinary differential equations* will expect this new booklet to be concise, clear and informative ; they will not be disappointed. In a very few pages the author has contrived to give the non-specialist University student everything he will require for the explicit solution of the usual types of ordinary differential equations. There are six chapters : Equations of the first order and degree, Integral curves, Equations of higher degree, Equations of the second and higher orders, Linear equations, Solution in series.

Points worthy of note are : the prominence given to integral curves, with an excellent paragraph on singular points ; the really amazing amount of material in the final chapter, where in twenty pages the method of solution in series is developed and applied to the hypergeometric series, the Legendre equation and the Bessel equation ; the 300 examples collected together at the end of the book, so that the student has to use his intelligence to decide what method to adopt. It should be added that these examples are roughly in the order of the bookwork, and that answers are given.

In spite of compression, or perhaps because of it, the exposition is clear and crisp. The only congested area is the final chapter ; bearing in mind its contents as described above, this is not surprising. Dr. Ince's opinion on any matter of exposition must not be opposed without considerable diffidence ; but it seems a pity that in solving simultaneous linear equations with constant coefficients he determines the proper arbitrary constants by the loathsome process of substituting in the original equations and examining the resulting set of linear algebraic equations. The reviewer's experience is that this method increases the likelihood of error ; but no doubt Dr. Ince has sound reasons for not using Routh's method.

Dr. Ince is to be congratulated on this excellent little book, which is bound to be popular. But he cannot have all the praise ; a word must be given to Messrs. Oliver and Boyd, who have provided good paper, clear type and a neat binding at the economical price of 4s. 6d.

T. A. A. B.

The relative difficulty of certain topics in mathematics for slow-moving ninth grade pupils. By V. S. MALLORY. Pp. x, 179. $2.10. 1939. (Teachers College, Columbia University, New York)

Recognition of individual differences among junior pupils is now almost a commonplace in educational thinking. Special classes are provided for backward children ; diagnostic tests are in use in some places, and remedial measures include suitable activities and individual adaptation of drill. Few such adjustments have, however, as yet been attempted for secondary pupils. It is still too often assumed that progress in secondary schools can be made at a uniform rate ; and that those who prove slower should at once be excluded and condemned to more practical work. This book is at once a plea for slow-moving pupils, and a guide to the material which has proved suitable to their needs.

Its findings are based on teaching and testing over a period of years. 511 pupils of I. Q. less than 110 took part in the final experiment; and the mass of evidence obtained from tests of intelligence and from measurement of progress was analysed to obtain answers to questions such as: Are these pupils deficient in ability to compute and in ability to reason in arithmetic situations? Does a year's study of general mathematics tend to improve the abilities of these pupils in computation and in reasoning in arithmetic problems? Is the I. Q. a factor? Can these pupils succeed with certain basic topics usually taught in first year algebra, such as making and evaluating formulae, using signed numbers, solving simple equations, and using general numbers in the fundamental operations? Is it true that these pupils like and enjoy work in other subjects better than they like work in mathematics?

Detailed reproduction of the findings cannot be given here; but the whole study is marked by sympathy for pupils and understanding of their interests and needs. The information it contains is well worthy of consideration by all mathematics teachers in the lower stages of the secondary school. One could wish that more particulars had been given of the activities and methods which proved interesting to these slow-moving pupils; but the book as a whole is a challenge to our impatience and a reminder that, even in the study of academic mathematics, interest and enthusiasm can be aroused in many children whom we have too readily labelled as dull—provided that the teacher takes pains to refrain from damaging their self-respect, takes trouble to understand them, and gives them time to move forward at a pace suited to the stage they have reached. C. M. F.

Arithmetic. By R. T. HUGHES. Pp. xv, 436. Without answers, 5s.; with answers, 5s. 6d. In parts: I, II, III without answers, each 2s. 3d.; with answers, each 2s. 6d.; I and II without answers, 4s.; with answers, 4s. 6d.; II and III without answers, 4s.; with answers, 4s. 6d. 1939. The London Mathematical series. (University of London Press)

Many teachers will welcome this book, which is the fruit of many years of teaching. It contains features which few "complete" school arithmetics, if any, contain in their entirety. Many of these are topics which are collectively referred to as the Arithmetic of Citizenship. While arithmetics on these topics have been written, these are by no means general school arithmetic books. There is no doubt that the importance of these topics is becoming increasingly recognised, and the book by Mr. Hughes supplies a real need. The topics are introduced throughout the general treatment and include Domestic (the housewife, shopping, the husband), Post Office, Wages and National Insurance, Trades (decorator, furnisher, bricklayer, carpenter), Business and Commerce, Agriculture (milk, crops, stock, poultry, market gardening), Town and County Administration (rates), Rent, House Purchase (mortgages, building societies), Gas, Electricity and Coal, Foods, Insurance (life, endowment), National Finance and Trade.

Apart from these topics, there are numerous illustrations from ordinary experience and the author has successfully carried out his policy of "bringing the pupil into effective contact with the kind of calculations and numerical thinking which arise in everyday life". This treatment is superimposed on a full school course in arithmetic, and the usual school course receives complete treatment. Some topics such as proportion in its many aspects have unusually thorough treatment, and there is a large number of problems of all grades of difficulty so that even the candidate for the Executive Class of the Civil Service will find everything he needs.

An unusual feature is the absence of the explanatory matter and worked examples which usually precede exercises. Mr. Hughes has been content to leave this to the teacher. There are, on the other hand, numerous tabloid explanations among the sets of examples and many are marked for discussion. While this method is suitable for the experienced teacher, there is not sufficient guidance for the younger member of the profession or for the student doing private study who would like to know the best way of working routine exercises such as multiplication of money and decimals and the tabulation of logarithmic calculations. Also, tabloid explanations are not sufficient for dealing with such subjects as stocks and shares. S. I.

First course in mathematics for technical students. By P. J. HALER and A. H. STUART. 3rd edition. Pp. vii, 295. 3s. 1939. (University Tutorial Press)

In this third edition, the authors have added chapters on logarithms and trigonometry. It will therefore be clear that the book has been written to meet the needs of junior students. Even so, it is surprising to meet with examples on addition and multiplication of decimals. The authors state in their preface that they have endeavoured to ensure that the atmosphere of the workshop should pervade the whole book. In this aim they have been largely successful, for many of the examples have some reference to workshop or technical topics.

In the treatment of the explanatory parts, however, the authors have not been so successful. The bias of the book provides excellent opportunities for dealing with fundamental ideas, such as the formula and functionality, and on the whole the authors have failed to make good use of these opportunities. Thus in the algebra section, while they are content to spend some time on manipulation processes, equations up to the quadratic are disposed of in some six pages, the graphical treatment is devoid of any important idea, and the treatment of the formula consists mostly of substitutions. S. I.

Seven-figure Trigonometrical Tables for Every Second of Time. Prepared by H.M. Nautical Almanac Office. Pp. 102. 10s. 1939. (H.M. Stationery Office)

Many astronomical angles, such as right ascension or hour angle, are measured in time units at the rate of 24 hours to 360°. Ephemerides give values of these quantities from day to day, or from hour to hour, and when working with these ephemerides it is almost invariably necessary to find trigonometrical ratios for these angles. Three- and four-figure tables for use by astronomers and others have been available for some years, but for greater accuracy it has hitherto been necessary first to convert to degrees, minutes and seconds of arc or to degrees and decimals, thus increasing the labour and chance of error. The tables now reviewed will prove extremely useful to those, mainly astronomers and surveyors, who need more than four-figure accuracy.

The tables, which form a very compact volume, are arranged in the convenient semi-quadrantal form and give, on each page, the four functions sine, tangent, cotangent and cosine for 120 values of the argument, covering 2 minutes of time. Linear interpolation is possible throughout, except for certain columns of cotangents with argument less than 15^m; a table of $x^s \cot x^s$, which can be interpolated linearly, provides a method of obtaining full 7-figure accuracy in the cotangent for arguments up to 30^m. To assist interpolation the range of differences occurring in each column is given at the head and foot of that column.

Two small criticisms occur to the writer. First, no indication is given as to which columns of cotangents cannot be interpolated linearly; it would

have been better to have omitted altogether the differences at the head and foot of these columns. Secondly, the cotangent can be obtained throughout with approximately seven-figure accuracy, except for angles between 30^m and 40^m, although interpolation would still have been linear with an extra figure in this range.

The precautions taken to guard against error appear to be adequate and the errata slip itself helps to inspire confidence since it suggests that proof-reading efforts have been concentrated on the tabular values. J. C. P. MILLER.

Introduction to Geometry. By A. W. SIDDONS and K. S. SNELL. Pp. vii, 167. 2s. 3d. 1939. Separately published, **Notes and Answers to Exercises in "Introduction to Geometry".** Pp. 24. 1s. 1939. (Cambridge University Press)

This book provides a preliminary course introductory to *A New Geometry* by the same authors reviewed in the *Mathematical Gazette* of last February.

The twenty short chapters consist largely of carefully graded exercises which, by encouraging observation, intuition, clear thinking and reasonably accurate drawing, lead the pupil almost unconsciously to an understanding of geometrical language, the acquisition of the simplest geometrical facts, the neat representation on paper of three-dimensional figures and the power to apply known facts to the discovery of others, though "no formal writing out of deductive work is attempted".

There follow five pages of Revision Papers, a list of facts, an Index and a page of coloured diagrams which includes three dissection verifications of Pythagoras' Theorem.

The whole forms an admirable first year's course of Geometry which will not only interest a beginner but will stimulate him to personal effort and accurate thought.

A few trifling errors have been overlooked:

On p. 17. Line 7 from bottom. For "making" read "marking".

On p. 51. Top line. For "Ex. 40" read "Ex. 41".

Perhaps it would be worth while to redraw the figures on p. 16 so that the two opposite edges *FK* and *GH* of a cube do not appear to be in one straight line.

The *Notes and Answers* are for the teacher. They embody many helpful suggestions for the effective use of the book, stressing those points which experience has shown to be of primary importance, and give the answers to the many numerical exercises. W. J. D.

NOTTINGHAM AND EAST MIDLAND BRANCH.

ON 2nd December, 1939, a meeting was held at University College, Nottingham, to form a Nottingham and East Midland Branch. The response was very gratifying, as sixty people attended, including twenty full members of the Association. The chair was taken by Professor H. T. H. Piaggio.

Dr. Max Black gave an address on "The New Spirit in Mathematical Education in England and U.S.A.". He gave a survey of the important developments in mathematical education during the last seventy years. He contrasted the older theory of mental discipline with the newer theory that the ideas of mathematics are more important than the technique. His observations in America indicated the readiness there to experiment and to carry ideas to their logical conclusion, even though with some danger of over-emphasis. This was illustrated by references to American textbooks.

Several members contributed to the lively discussion which followed the lecture. G. F. P. TRUBRIDGE, Hon. Sec.

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